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A.I. MARKUSHEVICH

AREAS AND LOGARITHMS

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Areas and Logarithms

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

А. И. Маркушевич

ПЛОЩАДИ И ЛОГАРИФМЫ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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AREAS AND LOGARITHMS

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FOREWORD

I first presented the lecture "Areas and Logarithms" in the autumn of 1951 at Moscow University to a large audience of senior schoolchildren intending to participate in the Mathematics Olympiad. The aim of the lecture was to present a geometric theory of logarithm, in which logarithms are introduced as various areas, with all the properties of the logarithms then being derived from those of the areas. The lecture also introduced the most simple concepts and elements of integral calculus, without resort to the concept of a derivative.

The lecture is published in this booklet with some additions. The reader can begin the book without even knowing what a logarithm is. He need only have a preliminary knowledge of the simplest functions and their graphical representation, of geometric progressions, and of the concept of limit.

If the reader wishes to broaden his knowledge of logarithms he is referred to the books *The Origin of Logarithms* by I. B. Abelson and *Series* by A. I. Markushevich (in Russian). The last chapter of the latter book contains an alternative theory of logarithms to that presented here.

The present publication includes a Supplement in which Simpson's rule and some of its applications can be found. In particular, the number π is calculated.

The author

1. Suppose a function is given which means that a method is indicated which allows us to associate every value of x with a corresponding value of y (the value of the function). Usually functions are defined by formulas. For instance, the formula $y = x^2$ defines y as a function of x. Here, for every number x (say, $x = 3$) the corresponding value of y is obtained by squaring the number x ($y = 9$). The formula $y = 1/x$ defines another function. Here for every nonzero x the corresponding value of y is the number inverse to x; if $x = 2$ then $y = 1/2$ and if $x = -1/2$ then $y = -2$.

When we speak of a function without indicating what particular function is meant we write $y = f(x)$ (read "y is f of x"). This means that y is a function of x (perhaps $y = x^2$, or $y = 1/x$, or some other function). Recall the idea of number lettering: the method just described closely resembles it, for we can speak either of the numbers 2, $-1/2$, $\sqrt{2}$ or of a number a, understanding it as one of these or any other number. Just as we use different letters to designate numbers, so we can describe a function as $y = f(x)$, or use some other notation, for instance $y = g(x)$, or $y = h(x)$, etc. Thus, if a problem involves two functions, we can denote one of them as $y = f(x)$ and the other as $y = g(x)$, and so on.

The function $y = f(x)$ can be shown graphically. To do this we take two mutually perpendicular straight lines Ox and Oy – the *coordinate axes* (see Fig. 1) – and, after choosing the scale unit, mark off the values of x on the x-axis and the corresponding values of $y = f(x)$ on the lines perpendicular to Ox (in the xOy plane). In so doing the rule of signs must be adhered to: positive numbers are denoted by line segments marked off to the right (along the x-axis) or upwards (from the x-axis) and negative numbers are marked off to the left or downwards. Note that the segments marked off from the point O along the *x-axis* are called *abscissas* and the segments marked off from Ox at right angles to it are called *ordinates*.

When the construction just described is carried out for all possible values of x, the ends of the ordinates will describe a curve in the plane which is the graph of the function $y = f(x)$ (in the case of $y = x^2$ the graph will be a parabola; it is shown in Fig. 2).

Take any two points A and B on the graph (Fig. 1) and drop

from them perpendiculars AC and BD to the x-axis. We obtain a figure $ACDB$; such a figure is called a *curvilinear trapezoid.* If, in a special case, the arc AB is a line segment not parallel to Ox, then an ordinary right-angled trapezoid is obtained. And if AB is a line segment parallel to Ox, then the resulting figure is a rectangle.

Thus, a right-angled trapezoid and a rectangle are special cases of a curvilinear trapezoid.

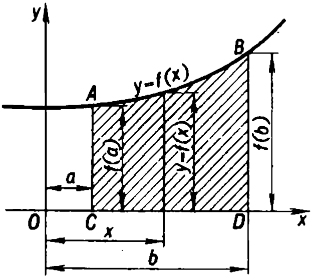

Fig. 1

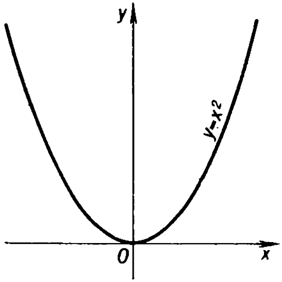

Fig. 2

The graph of the function depicted in Fig. 1 is located above the x-axis. Such a location is possible only when the values of the function are positive numbers.

In the case of negative values of the function the graph is located below the x-axis (Fig. 3). We then agree to assign a minus sign to the area of the curvilinear trapezoid and to consider it as negative.

Finally it is possible for the function to have different signs for the different intervals of the variation of x. Its graph is then located partly above Ox and partly below it; such a case is shown in Fig. 4. Here the area $A'C'D'B'$ of the curvilinear trapezoid must be considered *negative* and the area $A''C''D''B''$ *positive.* If in this case we take points A and B on the graph, as shown in the figure, and drop perpendiculars AC and BD from them to the x-axis, we then obtain a figure between these perpendiculars which is hatched in Fig. 4. The figure is called a curvilinear trapezoid, as before; it is bounded by the arc $AKA'B'LA''B''B$, two ordinates AC and BD and a segment CD of the abscissa axis. We take as its area the sum of the areas of the figures ACK, $KA'B'L$ and $LA''B''BD$, the areas of the first and the third of them being positive and the area of the second negative.

The reader will readily understand that under these conditions the area of the whole curvilinear trapezoid $ACDB$ may turn out

to be either positive or negative, or in some cases equal to zero. For instance, the graph of the function

$$y = ax \quad (a > 0)$$

is a straight line; here the area of the figure $ACDB$ (Fig. 5) is positive for $OD > OC$, negative for $OD < OC$, and equal to zero in the case of $OD = OC$.

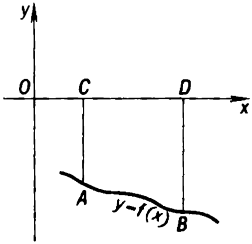

Fig. 3

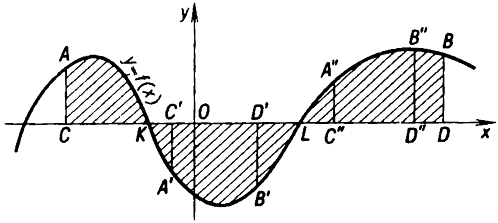

Fig. 4

2. Let us determine the area S of a curvilinear trapezoid. The need to calculate areas arises so often in various problems of mathematics, physics and mechanics that there exists a special science – integral calculus – devoted to methods of solving problems of this kind. We shall begin with a general outline of the solution of the problem, dividing the solution into two parts. In the first part we shall seek approximate values of the area, trying to make the error in the approximation infinitely small; in the second part we shall pass from the approximate values of the area to the exact value.

First let us replace the curvilinear trapezoid $ACDB$ by a stepped figure of the type shown in Fig. 6 (the figure is hatched). It is easy to calculate the area of the stepped figure: it is equal to the sum of the areas of the rectangles. This sum will be considered as being the approximate value of the sought area S.

When replacing S by the area of the stepped figure we make an error α; the error is made up of the areas of the curvilinear triangles blacked-out in Fig. 6. To estimate the error let us choose the widest rectangle and extend it so that its altitude becomes equal to the greatest value of the function (equal to BD in the case of Fig. 6). Next let us move all the curvilinear triangles parallel to the x-axis so that they fit into that rectangle; they will form a toothed figure resembling the edge of a saw (Fig. 7). Since the whole figure fits into the rectangle, the error α

equal to the sum of the areas of the saw teeth* must be less than the area of the rectangle. If its base is δ we obtain $|\alpha| < \delta \cdot BD$. Hence, the error α can be made infinitesimal if the rectangles in Fig. 6 are taken to be so narrow that the base δ of the widest of them is a sufficiently small number. For example, if $BD = 20$, and we want the area of the stepped figure to differ from S by less than 0.001, we must assume $\delta \cdot BD = 20\delta$ to be less than

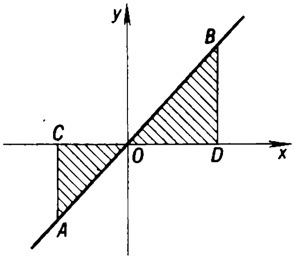

Fig. 5

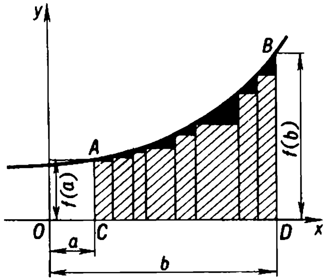

Fig. 6

0.001, i.e. $\delta < 0.00005$. Then

$$|\alpha| < \delta \cdot BD < 0.001.$$

Nevertheless, however small we make δ, an error α will result every time, if only a very small one, since the area of the curvilinear trapezoid does not equal that of the stepped figure.

The second, and final, part of the solution of the problem consists in passing to the limit. We assume that we consider not one, and not two, but an infinite number of stepped figures such as the one shown in Fig. 6. We take more and more rectangles, increasing their number indefinitely, making the base δ of the widest rectangle smaller and smaller, so that it tends to zero. The error α resulting from the replacement of the area of the curvilinear trapezoid by the area of the stepped figure will

In Figs. 6 and 7 the graph of the function is like the slope up (or down) a hill. Were the graph more complicated, with alternating rises and descents (see, for instance, Fig. 4), then the curvilinear triangles transferred into a single rectangle would overlap and the sum of their areas could turn out to be larger than the area of the rectangle. To make our reasoning applicable to this more complex case as well, let us divide the figure into parts to make the graph, within the limits of a single part, look like a single rise, or a single descent, and consider each part separately.

then become increasingly small, and indefinitely approach zero as well. The required area S will be obtained as the limit of the areas of the stepped figures.

3. Let us follow the reasoning used in Section 2 to estimate the area of the curvilinear trapezoid in the very important special case when the function $y = f(x)$ is a power with an integral nonnegative exponent $y = x^k$. For the exponents $k = 0, 1, 2$, we obtain the functions $y = x^0 = 1$, $y = x^1 = x$, $y = x^2$. Their graphs

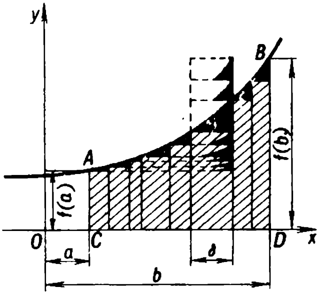

Fig. 7

are easy to construct: they are, respectively, a straight line parallel to the x-axis and passing above Ox at a unit distance (Fig. 8), a bisector of the angle xOy (Fig. 9), and a parabola (Fig. 10).

If we use higher exponents, we obtain the functions $y = x^3$, $y = x^4$, $y = x^5$. These graphs are shown in Figs. 11, 12 and 13.

If k is an odd number, the graphs are symmetrical with respect to the point O (Figs. 9, 11, 13), and if k is an even number, then they are symmetrical with respect to the y-axis (Figs. 8, 10, 12).

If $k \geqslant 1$, the graphs pass through the point O. In this case the greater the value of k, the closer the graphs are to the x-axis in the proximity of the point O and the steeper they rise upwards (or fall downwards) as they recede from the point O.

Each of the figures 8-13 contains a hatched curvilinear trapezoid. The areas of these trapezoids are easy to find when $k = 0$ and $k = 1$. Indeed, if $k = 0$, the area of $ACDB$ is equal to $CD \cdot AC = (b - a) \cdot 1 = b - a$; if $k = 1$, then the area of $ACDB$ is

$$CD \cdot \frac{AC + BD}{2} = (b - a)\frac{a + b}{2} = \frac{b^2 - a^2}{2}.$$

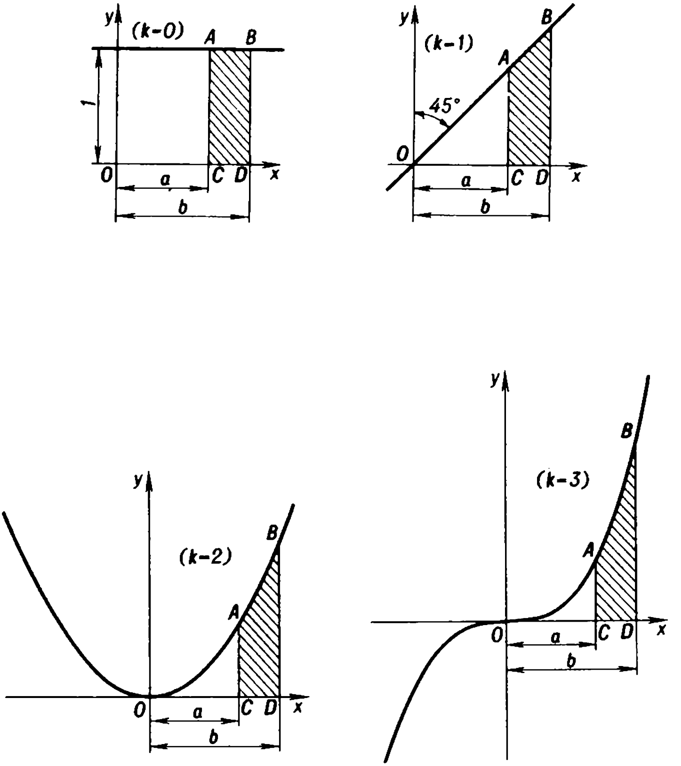

Fig. 11

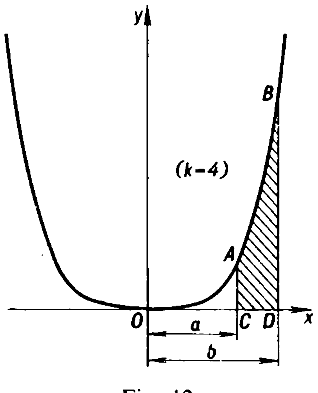

Fig. 12

Let us prove that if $k = 2$, the area of $ACDB$ is equal to $\frac{b^3 - a^3}{3}$; if $k = 3$, the area of $ACDB$ is $\frac{b^4 - a^4}{4}$, etc. We shall prove that in the general case, when k is any integral nonnegative number, the area of the corresponding curvilinear trapezoid is equal to $\frac{b^{k+1} - a^{k+1}}{k + 1}$. It is evident that this general result covers all the special cases discussed above.

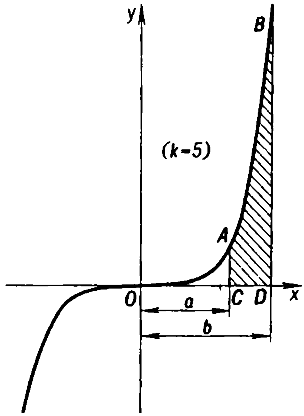

Fig. 13

To make it easier to follow the calculations below let us assume a definite numerical value for the exponent k, say $k = 5$. Let us further suppose that $0 < a < b$. Consequently we refer to the graph of the function $y = x^5$ and, following the reasoning outlined in Section 2, prove that the area of the curvilinear trapezoid $ACDB$ (Fig. 14) is equal to $\frac{b^6 - a^6}{6}$.

4. We have to calculate the sum of the areas of the very large number of rectangles which make up the stepped figure (Fig. 14). To simplify our job we choose the rectangles so that their areas form a geometric progression. To do this we take the points E, F, G, H, ..., I on the x-axis so as to make the line segments $OC = a$, OE, OF, OG, ..., OI, $OD = b$ form a geometric progression: we designate the number of terms in this progression by $n + 1$ and its common ratio by q (since

$b > a$, $q > 1$). Then we have the equalities

$$OC = a, \quad OE = aq, \quad OF = aq^2,$$

$$OG = aq^3, \ \ldots, \ OI = aq^{n-1}, \quad OD = aq^n = b.$$

Fig. 14 depicts six rectangles and hence $n + 1 = 7$, but we shall suppose in what follows that n is an arbitrarily large number, say $n = 1000$, $10\,000$, $100\,000$, etc.

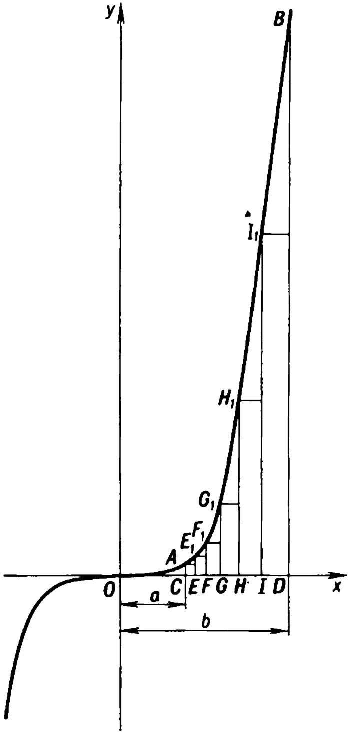

Fig. 14

The bases of the rectangles form a geometric progression with the same common ratio q:

$$CE = OE - OC = a(q - 1), \quad EF = OF - OE = aq(q - 1),$$

$$FG = OG - OF = aq^2(q - 1), \ \ldots,$$

$$ID = OD - OI = aq^{n-1}(q - 1)$$

(the number of terms in this and the following progressions is equal to n and not to $n+1$).

The altitudes of the rectangles are the ordinates CA, EE_1, FF_1, GG_1, ..., II_1; each of them is equal to the fifth power of the abscissa corresponding to it (remember that we have assumed $y = x^5$). Hence

$$CA = OC^5 = a^5, \quad EE_1 = OE^5 = a^5q^5, \quad FF_1 = OF^5 = a^5q^{10},$$
$$GG_1 = a^5q^{15}, \ \ldots, \ II_1 = OI^5 = a^5q^{5(n-1)}.$$

We see that the altitudes of the rectangles also form a geometric progression with the common ratio q^5 $(= q^k)$.

Since the bases of the rectangles form a progression with the common ratio q and the altitudes form a progression with the common ratio q^5 $(= q^k)$, the areas of the rectangles must form a progression with the common ratio $qq^5 = q^6$ $(= q^{k+1})$:

$$CE \cdot CA = a(q-1)a^5 = a^6(q-1);$$
$$EF \cdot EE_1 = aq(q-1)a^5q^5 = a^6q^6(q-1);$$
$$FG \cdot FF_1 = aq^2(q-1)a^5q^{10} = a^6q^{12}(q-1);$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$ID \cdot II_1 = aq^{n-1}(q-1)a^5q^{5(n-1)} = a^6q^{6(n-1)}(q-1).$$

Therefore the sum of the areas of the rectangles equal to the area of the stepped figure is the sum of the geometric progression with the first term $a^6(q-1)$, the last term $a^6q^{6(n-1)}(q-1)$ and the common ratio q^6:

$$\frac{a^6q^{6(n-1)}(q-1)q^6 - a^6(q-1)}{q^6-1} = [(aq^n)^6 - a^6]\frac{q-1}{q^6-1} =$$
$$= \frac{b^6 - a^6}{q^5+q^4+q^3+q^2+q+1}$$

$\left(\text{we have used the equalities } b = aq^n \text{ and } \frac{q^6-1}{q-1} = q^5 + q^4 + \right.$
$\left. + q^3 + q^2 + q + 1\right)$.

5. Let us increase indefinitely the number of the rectangles n. Since the bases of the rectangles form an increasing geometric progression ($q > 1$), the first one must be the smallest. But the sum of the lengths of all the n bases is equal to $b - a$; therefore

the length of CE is less than $\frac{b-a}{n}$, i. e. $aq - a < \frac{b-a}{n}$, whence $q - 1 < \frac{b-a}{na}$.

The right-hand side of the last inequality tends to zero when n increases indefinitely; since the left-hand side is positive it must tend to zero too, that is, q tends to unity.

This implies that q^2, q^3, q^4 and q^5 also tend to unity, the sum $q^5 + q^4 + q^3 + q^2 + q + 1$ tends to $1+1+1+1+1+1 = 6$ and, hence, the whole area of the stepped figure, equal to

$$\frac{b^6 - a^6}{q^5 + q^4 + q^3 + q^2 + q + 1},$$

tends to the limit

$$\frac{b^6 - a^6}{6}.$$

The required area of the curvilinear trapezoid must equal that particular limit:

$$S = \frac{b^6 - a^6}{6}.$$

We have obtained this result for $k = 5$. If we were to make these calculations in the general case for any natural k, we would obtain

$$S = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

Thus we have proved that the area of the curvilinear trapezoid bounded above by the arc of thc graph of the function $y = x^k$ and located between two ordinates with the abscissas a and b is equal to $\frac{b^{k+1} - a^{k+1}}{k+1}$.

6. We obtained the results of the previous section assuming that $0 < a < b$, i. e. that the curvilinear trapezoid lies to the right of the y-axis. If $a < b < 0$, the proof is carried out in the same way. However, assuming the common ratio q of the progression to be positive and greater than unity as before, we must now take b as the first and a as the last term of the progression (since $|a| > |b|$). Repeating the computations we receive the same result:

$$S = \frac{b^{k+1} - a^{k+1}}{k+1}.$$

If k is an odd number, then $k+1$ is even and, hence, b^{k+1} and a^{k+1} are positive numbers, the first number being less than the second. Therefore, in this case S is expressed by a negative number. This is clearly to be expected, since for odd k the corresponding curvilinear trapezoid lies below the x-axis (see the left-hand parts of Figs. 11 and 13).

Let us return to the case $0 < a < b$. If we consider b to be invariable and make a tend to zero, then the curvilinear trapezoid will extend to the left, and for a equal to zero will turn into a *curvilinear triangle OBD* (Fig. 15) (we assume that $k \geqslant 1$). It is obvious that for a tending to zero the area of the

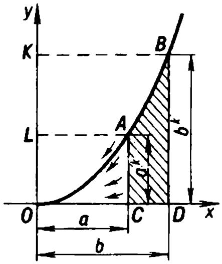

Fig. 15

curvilinear trapezoid will tend to the area of the curvilinear triangle. Indeed, the difference between the second and the first area will be less than the area $OCAL$, which itself tends to zero. On the other hand, for a tending to zero, the area of the curvilinear trapezoid tends to $\frac{b^{k+1}}{k+1}$, as is seen from the obtained formula. Therefore the area of the curvilinear triangle ODB is equal to $\frac{b^{k+1}}{k+1}$, i. e. it is $k+1$ times less than the area of the rectangle $ODBK$, or, the same thing, $k+1$ times less than the product of the "legs of the right-angled triangle" ODB (we have put inverted commas since we speak here not of the ordinary triangle, but of the curvilinear one). For $k = 1$ we have a function $y = x$, with the graph becoming a straight line (see Fig. 9) and the triangle an ordinary right triangle with its area equalling $\frac{1}{1+1} = \frac{1}{2}$ of the product of the legs.

We obtain analogous results if we proceed from the assumption $a < b < 0$ (the curvilinear trapezoid is located to the left of Oy). Taking a to be invariable we make b tend to zero; then the expression $\frac{b^{k+1} - a^{k+1}}{k+1}$ tends to the limit $-\frac{a^{k+1}}{k+1}$. This will be exactly the area of the corresponding curvilinear triangle.

A curvilinear triangle can be considered as a special case of a curvilinear trapezoid. From what we have established it follows that the formula

$$S = \frac{b^{k+1} - a^{k+1}}{k+1}$$

remains valid for a curvilinear triangle as well. We need only put in it $a = 0$ (if the triangle lies to the right of Oy) or $b = 0$ (if the triangle lies to the left of Oy).

7. Let us return to the general problem of the areas of curvilinear trapezoids. Let $ACDB$ be a curvilinear trapezoid bounded by the arc

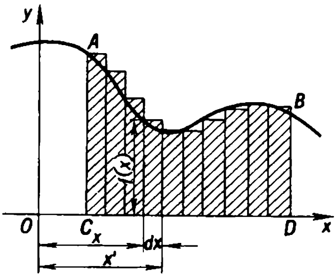

Fig. 16

AB of the graph of the function $y = f(x)$, two perpendiculars AC and BD dropped from the end points of the arc to the x-axis, and by the segment CD of the line Ox intercepted by the feet of the perpendiculars (Fig. 16). If $OC = a$ and $OD = b$ with $a < b$, then the area $ACDB$ is designated

$$\int_a^b f(x)\,dx. \qquad (*)$$

Every detail in this designation has a definite meaning. Here we have a function $f(x)$ whose graph is the upper part of the boundary of the curvilinear trapezoid, and also the numbers a and b,

specifying the right part and the left part of the boundary. The designation (∗) defines the method of seeking the area of $ACDB$; this method was given in Sections 2 and 3 and involves calculating the sum of the areas of the rectangles constituting the stepped figure and passing to the limit. The sign $\int$ is the extended letter S, the initial letter of the Latin word *summa*, sum. The unusual shape of the letter S implies that the calculation of the area of the curvilinear trapezoid involves not only summation, but must also include the passage to the limit. To the right of the sign $\int$, which is called the *integral* sign (from the Latin *integer* meaning full, entire), is the product $f(x)\,dx$. It represents the area of the rectangle with altitude $f(x)$ and base dx. The letter d standing before x is the initial letter of the Latin word *differentia* meaning difference; dx denotes the difference between the two values of x (see Fig. 16): $dx = x' - x$. The number a is the lower limit and b the upper limit of the integral (here the word "limit" means "boundary").

Thus, the designation (∗) for the area of the curvilinear trapezoid carries, on the one hand, all the information concerning its form and dimensions (given by the numbers a and b and by the function $f(x)$), and, on the other hand, contains within it the method of seeking the area of the trapezoid by calculating the areas of the rectangles with altitudes $f(x)$ and bases dx, by summing these areas and by passing to the limit (the integral sign shows that the summation and passing to the limit must be carried out). The designation (∗) should be read as "the integral from a to b, f of x dx". We repeat once again that this designation expresses the area of the curvilinear trapezoid $ACDB$. Using the new designation we may express the results of Section 5 as follows:

$$\int_a^b x^k\,dx = \frac{b^{k+1} - a^{k+1}}{k+1}$$

(k is an integral nonnegative number). The last equation should be read as "the integral from a to b of x^k dx is equal to the difference b^{k+1} and a^{k+1}, divided by $k + 1$".

8. Let us define some simple properties of integrals. Obviously the area $ACDB$ added to the area $BDD'B'$ yields the area $ACD'B'$ (Fig. 17). But the first area is equal to $\int_a^b f(x)\,dx$, the second to $\int_b^c f(x)\,dx$, and the third to $\int_a^c f(x)\,dx$;

therefore we have $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$. Here $a < b < c$; now if $a < c < b$ (Fig. 18), then, taking into account that the areas $ACD'B'$ and $B'D'DB$ together yield the area $ACDB$, we obtain

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

In introducing the concept of the integral $\int_a^b f(x)\,dx$ in Section 7 we assumed that $a < b$, i.e. that the lower limit is less than the

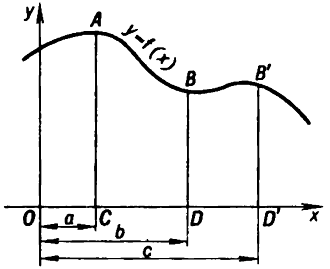

Fig. 17

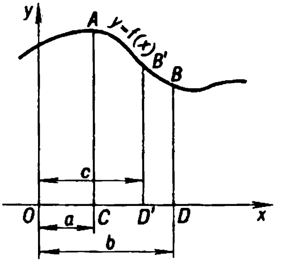

Fig. 18

upper limit. That is why the area $BDD'B'$, where $OD = b$ and $OD' = c$ for $b < c$ (Fig. 17), was written as $\int_b^c f(x)\,dx$ and for $b > c$ (Fig. 18) as $\int_c^b f(x)\,dx$ (every time the lower limit is less than the upper limit). In the first case the difference between the integrals $\int_a^c f(x)\,dx - \int_a^b f(x)\,dx$ was equal to $\int_b^c f(x)\,dx$, and in the second case to $\int_c^b f(x)\,dx$ (we have made use of the equalities of the integrals written above). To be able to express both cases by the same formula we shall agree that for $b > c$ we may write

$$\int_b^c f(x)\,dx = -\int_c^b f(x)\,dx.$$

In other words, we shall now include an integral whose lower limit is greater than the upper limit, meaning by this the area of a curvilinear trapezoid taken with the opposite sign. Then, instead of the two different formulas

$$\int_a^c f(x)\,dx - \int_a^b f(x)\,dx = \int_b^c f(x)\,dx \quad (b < c)$$

and

$$\int_a^c f(x)\,dx - \int_a^b f(x)\,dx = -\int_c^b f(x)\,dx \quad (b > c)$$

we shall write in all cases

$$\int_a^c f(x)\,dx - \int_a^b f(x)\,dx = \int_b^c f(x)\,dx \quad (b \neq c).$$

For $b = c$ the left-hand side vanishes; therefore it is legitimate to consider the integral $\int_b^b f(x)\,dx$, assuming it to be equal to zero.

Thus, irrespective of whether $b < c$, $b > c$ or $b = c$, we can use the formula

$$\int_a^c f(x)\,dx - \int_a^b f(x)\,dx = \int_b^c f(x)\,dx$$

in all cases. This formula can also be written as follows:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

We leave it to the reader to verify, with the results established in this Section, that the formula

$$\int_a^b x^k\,dx = \frac{b^{k+1} - a^{k+1}}{k+1}$$

is valid for *any* a and b (and not only for $0 \leqslant a < b$ or $a < b \leqslant 0$).

9. Assume that $f(x)$ is written as the sum or the difference of two functions: $f(x) = g(x) + h(x)$ or $f(x) = g(x) - h(x)$ (for example, $f(x) = x^3 - x^5$). Then the integral of $f(x)$ can be replaced by the sum or the difference of the integrals of the functions $g(x)$ and

$h(x)$:

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx + \int_a^b h(x)\,dx$$

$\left(\text{or } \int_a^b f(x)\,dx = \int_a^b g(x)\,dx - \int_a^b h(x)\,dx\right)$. For example,

$$\int_a^b (x^3 - x^5)\,dx = \int_a^b x^3\,dx - \int_a^b x^5\,dx = \frac{b^4 - a^4}{4} - \frac{b^6 - a^6}{6}.$$

Let us prove this property of integrals, taking the case of the sum. Let $f(x) = g(x) + h(x)$; the graphs of the three functions $g(x)$, $h(x)$ and $f(x)$ are depicted in Fig. 19. We have to prove that

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx + \int_a^b h(x)\,dx,$$

that is, that the area of $ACDB$ is equal to the sum of the areas of $A_1C_1D_1B_1$ and $A_2C_2D_2B_2$. Let us divide the segment of the x-axis between the points $x = a$ and $x = b$ into parts and construct the corresponding stepped figures for the three curvilinear trapezoids depicted in Fig. 19. It is evident that the area of each rectangle in the lower part of the figure is equal to the sum of the areas of the two rectangles shown in the two upper parts of the figure. Therefore the area of the lower stepped figure is equal to the sum of the areas of the two stepped figures lying above. This connection between the areas of the stepped figures will remain no matter how we divide the interval on the x-axis between $x = a$ and $x = b$. If this interval is divided into an indefinitely increasing number of parts whose lengths tend to zero, then the lower area will tend to the limit $\int_a^b f(x)\,dx$ and the two parts lying above will tend to the limits

$$\int_a^b g(x)\,dx \text{ and } \int_a^b h(x)\,dx.$$

Since the limit of the sum is equal to the sums of the limits,

$$\int_a^b f(x)\,dx = \int_a^b [g(x) + h(x)]\,dx = \int_a^b g(x)\,dx + \int_a^b h(x)\,dx;$$

and that is what we had to prove.

In just the same way it can be proved that

$$\int_a^b [g(x) - h(x)]\, dx = \int_a^b g(x)\, dx - \int_a^b h(x)\, dx.$$

It is easy to see that this property of integrals is also valid when $f(x)$ is the sum of the greater number of summands. For

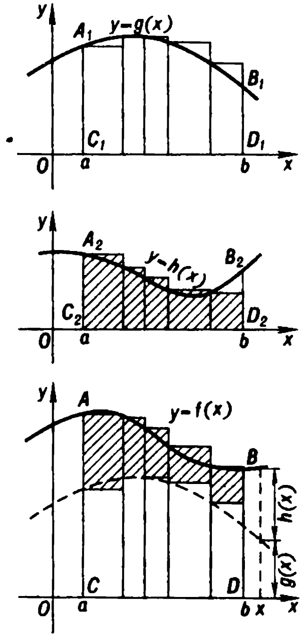

Fig. 19

example, if $f(x) = g(x) - h(x) + k(x)$, then

$$\int_a^b [g(x) - h(x) + k(x)]\, dx =$$

$$= \int_a^b [g(x) - h(x)]\, dx + \int_a^b k(x)\, dx =$$

$$= \int_a^b g(x)\, dx - \int_a^b h(x)\, dx + \int_a^b k(x)\, dx.$$

10. We have also to find the connection between the integrals

$$\int_a^b f(x)\,dx \text{ and } \int_a^b Cf(x)\,dx,$$

where C is some (constant) number; for instance, we have to find the connection between the integrals $\int_a^b x^3\,dx$ and $\int_a^b 2x^3\,dx$. Let us show that

$$\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx,$$

for example,

$$\int_a^b 2x^3\,dx = 2\int_a^b x^3\,dx = 2\,\frac{b^4 - a^4}{4} = \frac{b^4 - a^4}{2}.$$

To simplify the reasoning we shall give to C a definite numerical value, say, $C = \frac{1}{2}$. Now we have to compare the integrals

$$\int_a^b f(x)\,dx \text{ and } \int_a^b \frac{1}{2}f(x)\,dx.$$

Fig. 20 depicts the curvilinear trapezoids whose areas are represented by these integrals. Let us divide the segment of the x-axis between the points $x = a$ and $x = b$ into a number of parts and construct the corresponding stepped figures. It is easy to see that the area of each rectangle in the lower part of the figure is equal to half the area of the rectangle lying above it (from the upper part of the figure). Therefore the area of the lower stepped figure is half that of the upper stepped figure. Passing to the limit as we did in Section 9, we conclude that the whole area of the lower curvilinear trapezoid is also half the area of the upper curvilinear trapezoid:

$$\int_a^b \frac{1}{2}f(x)\,dx = \frac{1}{2}\int_a^b f(x)\,dx.$$

In our argument the number C was positive; if we assume C

to be negative, for instance, $C = -\frac{1}{2}$, then we have to replace Fig. 20 by Fig. 21.

Comparing now the area of $ACDB$ with that of $A''C''D''B''$ we find that here, besides the variation in the absolute value of the

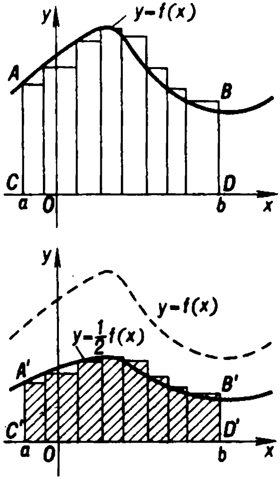

Fig. 20

area (its decrease by half), the sign changes too. Hence,

$$\int_a^b \left(-\frac{1}{2}\right) f(x)\,dx = -\frac{1}{2}\int_a^b f(x)\,dx.$$

It is clear that we assumed $C = \pm\frac{1}{2}$ only for clarity. In general, for any C, the equality

$$\int_a^b Cf(x)\,dx = C\int_a^b f(x)\,dx$$

is valid.

To show how the properties of integrals discussed in this and in the previous section can be used, let us compute the integral

$\int_0^1 (3x^2 - 2x + 1)\,dx$. We obtain

$$\int_0^1 (3x^2 - 2x + 1)\,dx = \int_0^1 3x^2\,dx - \int_0^1 2x\,dx + \int_0^1 1\,dx =$$

$$= 3\int_0^1 x^2\,dx - 2\int_0^1 x\,dx + \int_0^1 x^0\,dx =$$

$$= 3\,\frac{1^3 - 0^3}{3} - 2\,\frac{1^2 - 0^2}{2} + \frac{1^1 - 0^1}{1} = 3\cdot\frac{1}{3} - 2\cdot\frac{1}{2} + 1 = 1.$$

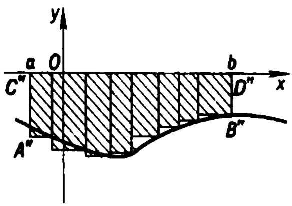

Fig. 21

11. Let us consider the function

$$y = x^{-1} = \frac{1}{x}.$$

Its graph is referred to as an *equilateral hyperbola*; it is shown in Fig. 22. If the formula for the area of the curvilinear trapezoid

$$\int_a^b x^k\,dx = \frac{b^{k+1} - a^{k+1}}{k+1},$$

derived earlier for the case $k \geqslant 0$, is applied to this case, then, noting that $k + 1 = 0$, $b^{k+1} = a^{k+1} = 1$, we obtain in the right-hand term the meaningless expression $\frac{0}{0}$. Hence our formula cannot be used in the case of $k = 1$.

Although the formula cannot be employed for the calculation of the integral $\int_a^b x^{-1}\,dx$, we can nevertheless study some properties of this integral.

We can prove if a and b are increased or decreased

the same number of times, i. e. are multiplied by the same $q > 0$, a new curvilinear trapezoid *with the same area* results. Needless to say we prove this property assuming that the curve whose arcs bound the curvilinear trapezoids from one side is an equilateral hyperbola and

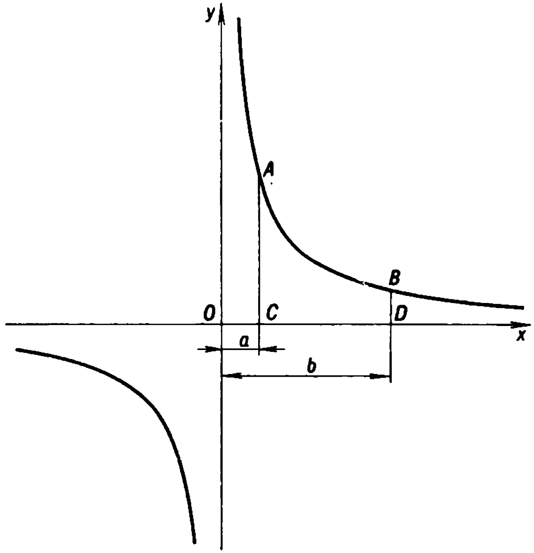

Fig. 22

not some other curve. In other words,

$$\int_{aq}^{bq} x^{-1}\,dx = \int_{a}^{b} x^{-1}\,dx,$$

for any $q\,(q > 0)$.

To make it easier to follow the proof let us give to q a definite numerical value, for example $q = 3$. Fig. 23 depicts two curvilinear trapezoids $ACDB$ and $A''C'D'B''$ corresponding to this case. The first trapezoid is narrower but higher, the second wider but lower. Let us prove that the increase in width in the second case is compensated for by the decrease in height, so that the area remains the same. To carry out the proof let us divide the first trapezoid into several narrower trapezoids and replace each of the latter trapezoids by a rectangle (Fig. 23). If the abscissa of each point of the stepped figure $ACDB$ constructed above is trebled and

the ordinates are left unchanged, a figure $A'C'D'B'$ results whose area is three times as large, since each rectangle becomes three times as wide. But the ends of the ordinates are no longer located on our hyperbola. Indeed, this hyperbola is the graph of the inverse proportion $y = \frac{1}{x}$, and to keep the points on the hyperbola the ordinate must be decreased by the same number of times as the abscissa is increased. If all the ordinates of the figure $A'C'D'B'$ are decreased three times the resulting figure is $A''C'D'B''$, which is a curvilinear trapezoid bounded above by the arc of the hyperbola $y = \frac{1}{x}$ and from the sides by the ordinates

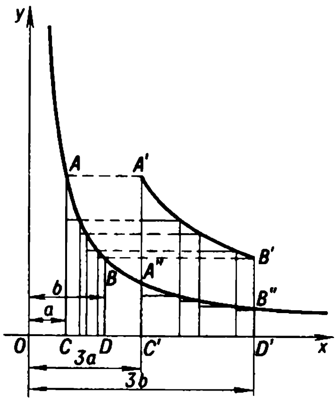

Fig. 23

constructed for $x = 3a$ and $x = 3b$. The resulting rectangles have bases three times as large as the original rectangles and altitudes three times as small. Hence their areas are the same as those of the original rectangles. Consequently, the areas of the two stepped figures are identical, as are their limits, i. e. the areas of the curvilinear trapezoids:

$$\int_{qa}^{qb} x^{-1}\,dx = \int_{a}^{b} x^{-1}\,dx.$$

We have proved this property assuming that $a < b$. But it is also true for $a = b$ and $a > b$. Indeed, if $a = b$, then $aq = bq$ and both integrals vanish, and hence the equality is not violated. If,

however, $a > b$, then $aq > bq$; in this case we have the equality

$$\int_{bq}^{aq} x^{-1}\,dx = \int_{b}^{a} x^{-1}\,dx$$

(now $b < a$ and therefore b and a exchange the roles they play). But we agreed in Section 8 that $\int_{a}^{b} f(x)\,dx$ means $-\int_{b}^{a} f(x)\,dx$ for $a > b$. Consequently

$$\int_{aq}^{bq} x^{-1}\,dx = -\int_{bq}^{aq} x^{-1}\,dx, \quad \int_{a}^{b} x^{-1}\,dx = -\int_{b}^{a} x^{-1}\,dx.$$

Since the right-hand sides of these relations are equal, the left-hand sides should be equal too:

$$\int_{aq}^{bq} x^{-1}\,dx = \int_{a}^{b} x^{-1}\,dx.$$

Thus, the relation we have proved remains valid irrespective of whether $a < b$, $a = b$ or $a > b$.

12. Now let us assume $a = 1$ and consider $\int_{1}^{b} x^{-1}\,dx$. If $b > 1$, the integral represents the area of the curvilinear trapezoid $ACDB$ (Fig. 24). Now if $b = 1$, it turns into zero, and, finally, if $0 < b < 1$, then the lower limit of the integral is less than the upper limit and we obtain

$$\int_{1}^{b} x^{-1}\,dx = -\int_{b}^{1} x^{-1}\,dx.$$

This means that in this case the integral differs from the area of the curvilinear trapezoid $B'D'CA$ only in its sign (Fig. 25). In any case, for every positive value of b the integral $\int_{1}^{b} x^{-1}\,dx$ has quite a definite value. It is positive when $b > 1$, equal to zero when $b = 1$ and negative for $b < 1$.

It is evident that the integral $\int_{1}^{b} x^{-1}\,dx$ is a function of b. This function plays a very important role in mathematics; it is

called a *natural logarithm* of the number b and is designated $\ln b$. Here l and n are the initial letters of the Latin words *logarithmus* – logarithm, and *naturalis* – natural. Thus,

$$\int_1^b x^{-1}\,dx = \ln b.$$

Let us discuss some properties of natural logarithms. First of all we have

$$\ln b > 0 \text{ if } b > 1;\ \ln 1 = 0;\ \ln b < 0 \text{ if } b < 1.$$

Next we derive the main property of a logarithm, the fact that *the logarithm of a product is equal to the sum of the logarithms*

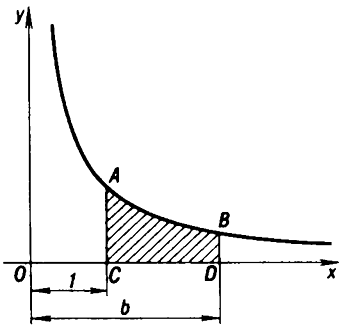

Fig. 24

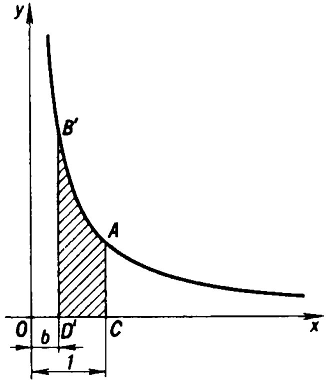

Fig. 25

of the factors, for instance, $\ln 6 = \ln 2 + \ln 3$. In the general case

$$\ln(bc) = \ln b + \ln c,$$

that is

$$\int_1^{bc} x^{-1}\,dx = \int_1^b x^{-1}\,dx + \int_1^c x^{-1}\,dx.$$

Indeed, according to what was proved above

$$\int_1^c x^{-1}\,dx = \int_q^{qc} x^{-1}\,dx$$

for any $q > 0$. Let us take $q = b$; then we shall have

$$\int_1^c x^{-1}\,dx = \int_b^{bc} x^{-1}\,dx.$$

Therefore

$$\int_1^b x^{-1}\,dx + \int_1^c x^{-1}\,dx = \int_1^b x^{-1}\,dx + \int_b^{bc} x^{-1}\,dx.$$

But the last sum can be replaced, according to the property derived in Section 8, by the integral $\int_1^{bc} x^{-1}\,dx$. Hence we have

$$\int_1^b x^{-1}\,dx + \int_1^c x^{-1}\,dx = \int_1^{bc} x^{-1}\,dx,$$

just what we had to prove.

This property allows us to derive some corollaries. Let $b > 0$; then, according to the property proved above,

$$\ln 1 = \ln\left(b\frac{1}{b}\right) = \ln b + \ln\frac{1}{b};$$

since $\ln 1 = 0$, $\ln b + \ln\frac{1}{b} = 0$, whence

$$\ln\frac{1}{b} = -\ln b.$$

For example, $\ln\frac{1}{2} = -\ln 2$. Next we have

$$\ln\frac{c}{b} = \ln\left(c\frac{1}{b}\right) = \ln c + \ln\frac{1}{b} = \ln c - \ln b$$

if $b > 0$ and $c > 0$; in other words, *a logarithm of a quotient is equal to the difference of the logarithms of the divident and the divisor.*

The main property of the logarithm was formulated for the product of two factors, but it is valid for the product of any number of the factors as well. Thus, for instance, if we have three factors, then we obtain

$$\ln(bcd) = \ln[(bc)d] = \ln(bc) + \ln d =$$
$$= (\ln b + \ln c) + \ln d = \ln b + \ln c + \ln d.$$

Evidently, whatever the number of the factors, the logarithm of their product is always equal to the sum of the logarithms of the factors.

Let us apply this property to the logarithm of a power with the

integral positive exponent k. We have

$$\ln b^k = \underbrace{\ln (bb \dots b)}_{k \text{ times}} = \underbrace{\ln b + \ln b + \dots + \ln b}_{k \text{ times}} = k \ln b.$$

For example, $\ln 16 = \ln 2^4 = 4 \ln 2$.

Let $c = \sqrt[k]{b}$; then $c^k = b$ and consequently

$$\ln b = \ln c^k = k \ln c = k \ln \sqrt[k]{b},$$

whence we have

$$\ln \sqrt[k]{b} = \frac{1}{k} \ln b.$$

For instance

$$\ln \sqrt[3]{2} = \frac{1}{3} \ln 2.$$

If $c = b^{\frac{p}{q}}$, where p and q are positive integers, then, by the properties proved above,

$$\ln b^{\frac{p}{q}} = \ln \sqrt[q]{b^p} = \frac{1}{q} \ln b^p = \frac{1}{q} \cdot p \ln b = \frac{p}{q} \ln b.$$

Hence, the property

$$\ln b^k = k \ln b$$

is valid not only when k is a positive integer, but also when k is a fraction in the form $\frac{p}{q}$.

It is easy to see that the same property is valid for a negative k as well (integral and fractional). Indeed, if $k < 0$, then $-k > 0$ and we have

$$\ln b^k = \ln \frac{1}{b^{-k}} = -\ln b^{-k} = -(-k \ln b) = k \ln b.$$

Finally, the same property is valid for $k = 0$ too:

$$\ln b^0 = \ln 1 = 0 = 0 \cdot \ln b.$$

Thus, for any rational k (positive, equal to zero or negative, integral or fractional) it is true that

$$\ln b^k = k \ln b.$$

It would also be possible to prove that this relation is true for an irrational k; for example,

$$\ln b^{\sqrt{2}} = \sqrt{2} \ln b.$$

We shall accept the latter statement without proof and shall use the following property: *the natural logarithm of a power is equal to the exponent multiplied by the natural logarithm of the base of the power*, for all possible values of the exponent k, both rational and irrational.

13. Let us compute some logarithms. Perhaps we should calculate $\ln 2$, i. e. find the area of the curvilinear trapezoid $ACDB$ depicted in Fig. 26*a*. We divide the line segment CD into ten equal parts and draw the respective ordinates: K_1L_1, K_2L_2, ..., K_9L_9. To find

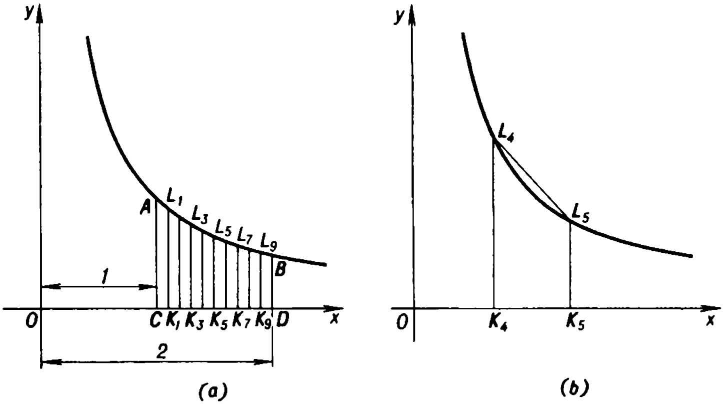

Fig. 26

the best approximation to $\ln 2$, we replace each of the ten resulting narrow curvilinear trapezoids not by a rectangle as we did before, but by an ordinary, i. e. rectilinear trapezoid. To do this, we connect point A with L_1, point L_1 with L_2, and so on, ..., L_9 with B by line segments. In Fig. 26*a* it is difficult to distinguish between ordinary and curvilinear trapezoids; to make the distinction more visible we increase Fig. 26*b* in scale. The area of each trapezoid is equal to the product of half the sum of the bases by the altitude; but in our case all the altitudes are equal:

$$CK_1 = K_1K_2 = \ldots = K_9D = 0.1.$$

Therefore, the areas of the trapezoids will be as follows:

$$\frac{AC + K_1L_1}{2}\cdot 0.1; \quad \frac{K_1L_1 + K_2L_2}{2}\cdot 0.1 \ldots; \quad \frac{K_9L_9 + BD}{2}\cdot 0{,}1;$$

the sum of these areas is equal to

$$0.1\,\frac{(AC + K_1L_1) + (K_1L_1 + K_2L_2) + \ldots + (K_9L_9 + BD)}{2},$$

or to

$$0.1\,(0.5\,AC + K_1L_1 + K_2L_2 + \ldots + K_9L_9 + 0.5\,BD).$$

It remains only to draw the reader's attention to the fact that all the bases of the trapezoids are the ordinates of the graph of the function $y = \frac{1}{x}$ corresponding to the following abscissas:

$$1;\ 1.1;\ 1.2;\ 1.3;\ 1.4;\ 1.5;\ 1.6;\ 1.7;\ 1.8;\ 1.9;\ 2.$$

Therefore

$$AC = \frac{1}{1} = 1.000;\ K_1L_1 = \frac{1}{1.1} = 0.909;\ K_2L_2 = \frac{1}{1.2} = 0.833;$$

$$K_3L_3 = \frac{1}{1.3} = 0.769;\ K_4L_4 = \frac{1}{1.4} = 0.714;\ K_5L_5 = \frac{1}{1.5} = 0.667;$$

$$K_6L_6 = \frac{1}{1.6} = 0.625;\ K_7L_7 = \frac{1}{1.7} = 0.588;\ K_8L_8 = \frac{1}{1.8} = 0.556;$$

$$K_9L_9 = \frac{1}{1.9} = 0.526; \qquad BD = \frac{1}{2} = 0.500.$$

Consequently, the sum of the areas of the trapezoids is equal to

$$0.1\,(0.500 + 0.909 + 0.833 + 0.769 + 0.714 + 0.667 + \\ + 0.625 + 0.588 + 0.556 + 0.526 + 0.250) = 0.6937.$$

If we look more carefully at Fig. 26 we shall see that the sum of the areas of these trapezoids gives a value somewhat greater than that of the area of the curvilinear trapezoid. This means that we have found an approximate value for $\ln 2$ exceeding the real value (a major approximation), i. e. that $\ln 2$ is somewhat less than 0.6937.

Later we shall acquaint ourselves with another method of calculating logarithms which will make it possible to obtain, in particular, $\ln 2$ with a higher degree of accuracy.

14. If the abscissas are laid off not from the point O but from the point C (Fig. 27) and the new abscissas are denoted by the letter t, then the connection between the new and the old abscissas of one and the same point will be

$$x = 1 + t.$$

This connection will be true for any point if we assume that $t > 0$ when $x > 1$, and $t \leqslant 0$ when $x \leqslant 1$. On substituting $1 + t$ for x the

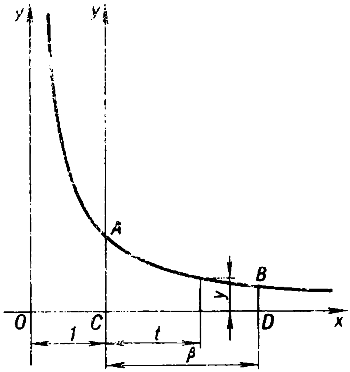

Fig. 27

function $y = \frac{1}{x}$ becomes $y = \frac{1}{1+t}$, but the graph remains the same. Anything new resulting from the introduction of t refers only to the origin (C instead of O), and, hence, to the new axis Cy (parallel to Oy); the curve itself remains unaltered. The area $ACDB$ does not change either. But when we took x as the abscissa this area was expressed by the integral

$$\int_1^{1+\beta} x^{-1}\,dx = \ln(1 + \beta)$$

(here $\beta = CD$).

Now when we take t for the abscissa, the same area is expressed by the integral $\int_0^\beta (1 + t)^{-1}\,dt$. Comparing the two integrals we obtain

$$\ln(1 + \beta) = \int_0^\beta (1 + t)^{-1}\,dt.$$

Let us now note the following identity:

$$1 - t + t^2 - t^3 + \ldots - t^{2n-1} = \frac{1 - t^{2n}}{1 + t}.$$

We can see it immediately if we note that the left-hand side contains a geometric progression with 1 as its first term, $-t$ as the common ratio and $-t^{2n-1}$ as its last term. This identity implies

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots - t^{2n-1} + \frac{t^{2n}}{1+t}.$$

Therefore

$$\ln(1+\beta) = \int_0^\beta \left[1 - t + t^2 - t^3 + \ldots - t^{2n-1} + \frac{t^{2n}}{1+t} \right] dt.$$

Now a more complex and cumbersome expression containing the sum of several summands has appeared under the integral sign instead of $(1+t)^{-1}$. We have learned by now that the integral of a sum or a difference of functions is equal to the sum or the difference of the integrals of these functions. Hence,

$$\ln(1+\beta) = \int_0^\beta 1\,dt - \int_0^\beta t\,dt + \int_0^\beta t^2\,dt - \int_0^\beta t^3\,dt + \ldots$$

$$\ldots - \int_0^\beta t^{2n-1}\,dt + \int_0^\beta \frac{t^{2n}}{1+t}\,dt.$$

We are now able to compute each of the integrals on the right-hand side except for the last integral. So we can write

$$\int_0^\beta 1\,dt = \beta, \quad \int_0^\beta t\,dt = \frac{\beta^2}{2}, \quad \int_0^\beta t^2\,dt = \frac{\beta^3}{3},$$

$$\int_0^\beta t^3\,dt = \frac{\beta^4}{4}, \ldots, \int_0^\beta t^{2n-1}\,dt = \frac{\beta^{2n}}{2n}.$$

It follows that

$$\ln(1+\beta) = \left(\beta - \frac{\beta^2}{2} + \frac{\beta^3}{3} - \frac{\beta^4}{4} + \ldots - \frac{\beta^{2n}}{2n}\right) + \int_0^\beta \frac{t^{2n}}{1+t}\,dt.$$

The expression in the parentheses on the right-hand side of the equation is a polynomial of degree $2n$ arranged according to the increasing powers of β. If the value of β is known, and if, in addition, n is taken to be a positive integer (it can be arbitrary), then the value of this polynomial can easily be computed. Difficulties arise only when we begin to compute the integral $\int\limits_0^\beta \frac{t^{2n}}{1+t}\,dt$. We shall prove that by making n sufficiently large we can make the integral arbitrarily small for $-1 < \beta \leqslant 1$. If this is so, then in computing $\ln(1+\beta)$ we can neglect the last integral; this will result in no more than a negligible error. Consequently we obtain the following approximate equality:

$$\ln(1+\beta) \approx \beta - \frac{\beta^2}{2} + \frac{\beta^3}{3} - \frac{\beta^4}{4} + \ldots - \frac{\beta^{2n}}{2n}.$$

15. To estimate the error of this approximate equality we must consider the deleted integral $\int\limits_0^\beta \frac{t^{2n}}{1+t}\,dt$. Assume, first, that $0 < \beta \leqslant 1$. Then, within the limits of integration, t remains positive and, consequently,

$$0 < \frac{t^{2n}}{1+t} < t^{2n}.$$

This means that the graph of the function $y = \frac{t^{2n}}{1+t}$ lies below that of the function $y = t^{2n}$ (Fig. 28); therefore, the area CBA_1 is less than the area CBA, i. e.

$$\int\limits_0^\beta \frac{t^{2n}}{1+t}\,dt < \int\limits_0^\beta t^{2n}\,dt = \frac{\beta^{2n+1}}{2n+1}.$$

Thus, the error of the approximate equation derived above is less than $\frac{\beta^{2n+1}}{2n+1}$; since $0 < \beta \leqslant 1$, this error can be made arbitrarily small when n is sufficiently large. Take, for instance,

$\beta = 1$; in this case the formula derived above will yield

$$\ln 2 \approx 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots + \frac{1}{2n-1} - \frac{1}{2n},$$

with an error less than $\frac{1}{2n+1}$. If we use this method to calculate $\ln 2$ with an accuracy to within 0.001, then we must assume $\frac{1}{2n+1}$ to be less than 0.001, i. e. $2n + 1 > 1000$; this condition

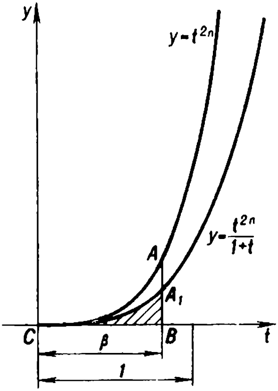

Fig. 28

can be satisfied by putting $2n = 1000$. But in this case the left-hand side of the equation will contain 1000 terms whose sum must be computed. It goes without saying that this is a difficult job. We shall soon learn how we can do it with the aid of another formula for $\ln 2$.

16. Let us again consider the integral $\int\limits_0^\beta \frac{t^{2n}}{1+t}\,dt$, but now putting $-1 < \beta < 0$. We know that

$$\int\limits_0^\beta \frac{t^{2n}dt}{1+t} = -\int\limits_\beta^0 \frac{t^{2n}\,dt}{1+t}.$$

The integral $\int\limits_{\beta}^{0}\frac{t^{2n}dt}{1+t}$ is equal to the area of the figure $ABCK$ shown hatched in Fig. 29. This figure is located above the line Ct, since $y=\frac{t^{2n}}{1+t}>0$ for $t>-1$. Consequently the area of $ABCK$ is a positive number, i. e. the integral $\int\limits_{\beta}^{0}\frac{t^{2n}dt}{1+t}$ is a positive

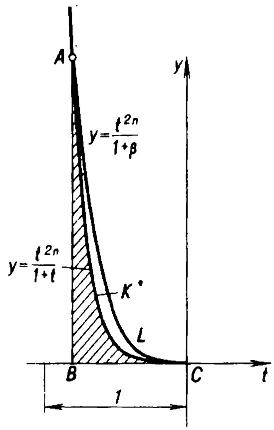

Fig. 29

number. It differs from the integral $\int\limits_{0}^{\beta}\frac{t^{2n}dt}{1+t}$ only in sign and is therefore equal to the absolute value of the latter:

$$\int\limits_{\beta}^{0}\frac{t^{2n}dt}{1+t}=\left|\int\limits_{0}^{\beta}\frac{t^{2n}dt}{1+t}\right|.$$

Further, let us note that for $t > \beta$ and $\beta > -1$ the inequality

$$1 + t > 1 + \beta > 0$$

holds, and hence,

$$\frac{1}{1+t} < \frac{1}{1+\beta}$$

and

$$\frac{t^{2n}}{1+t} < \frac{t^{2n}}{1+\beta}$$

This means that the graph of the function $y = \dfrac{t^{2n}}{1+t}$ lies below the graph of the function $y = \dfrac{t^{2n}}{1+\beta}$ in the interval $\beta < t < 0$ (Fig. 29). Therefore the area of the figure $ABCK$ is less than that of $ABCL$:

$$\int_{\beta}^{0} \frac{t^{2n}\,dt}{1+t} < \int_{\beta}^{0} \frac{t^{2n}\,dt}{1+\beta}.$$

It is easy to calculate the right-hand side of the inequality:

$$\int_{\beta}^{0} \frac{1}{1+\beta} t^{2n}\,dt = \frac{1}{1+\beta}\int_{\beta}^{0} t^{2n}\,dt =$$

$$= \frac{1}{1+\beta}\,\frac{0^{2n+1} - \beta^{2n+1}}{2n+1} = -\frac{\beta^{2n+1}}{(2n+1)(1+\beta)}$$

(this is a positive number since $\beta^{2n+1} < 0$, $1 + \beta > 0$ and $2n + 1 > 0$). It follows that

$$\left|\int_{0}^{\beta} \frac{t^{2n}}{1+t}\,dt\right| = \int_{\beta}^{0} \frac{t^{2n}\,dt}{1+t} < -\frac{\beta^{2n+1}}{(2n+1)(1+\beta)}.$$

And so by deleting the term $\displaystyle\int_{0}^{\beta} \frac{t^{2n}\,dt}{1+t}$ in the expression for $\ln(1+\beta)$ we make an error less in absolute value than

$-\dfrac{\beta^{2n+1}}{(2n+1)(1+\beta)}$ $(-1 < \beta < 0)$. It tends to zero for indefinitely increasing n.

Thus the approximate formula

$$\ln(1+\beta) \approx \beta - \frac{\beta^2}{2} + \frac{\beta^3}{3} - \frac{\beta^4}{4} + \ldots - \frac{\beta^{2n}}{2n}$$

is valid with an accuracy to within the factor $-\dfrac{\beta^{2n+1}}{(2n+1)(1+\beta)}$ for $-1 < \beta < 0$.

Let us make, for instance, $\beta = -\dfrac{1}{2}$; in this case the error of the approximation will be less than

$$\frac{1}{2^{2n+1}} : \left[\frac{1}{2}(2n+1)\right] = \frac{1}{(2n+1)\,2^{2n}}.$$

If we take $n = 4$, the last fraction will be equal to $\dfrac{1}{9 \cdot 2^8} = \dfrac{1}{9 \cdot 256} = \dfrac{1}{2304} < 0.0005$. This means that with this degree of accuracy we can write

$$\ln\frac{1}{2} \approx -\frac{1}{2} - \frac{1}{2^2 \cdot 2} - \frac{1}{2^3 \cdot 3} - \frac{1}{2^4 \cdot 4} - \frac{1}{2^5 \cdot 5} - \frac{1}{2^6 \cdot 6} - \frac{1}{2^7 \cdot 7} - \frac{1}{2^8 \cdot 8}.$$

The requisite calculations yield

$$\frac{1}{2} = 0.5000; \quad \frac{1}{2^2 \cdot 2} = 0.1250; \quad \frac{1}{2^3 \cdot 3} = 0.0417; \quad \frac{1}{2^4 \cdot 4} = 0.0156;$$

$$\frac{1}{2^5 \cdot 5} = 0.0062; \quad \frac{1}{2^6 \cdot 6} = 0.0026; \quad \frac{1}{2^7 \cdot 7} = 0.0011; \quad \frac{1}{2^8 \cdot 8} = 0.0005$$

and we obtain

$$\ln\frac{1}{2} \approx -0.6927 \approx -0.693$$

with an accuracy to within 0.001 (we take into account that the formula itself could involve an error of up to 0.0005, and, in addition, an error up to 0.00005 could arise when we reduce each of the eight summands to a decimal fraction).

Since $\ln\frac{1}{2} = -\ln 2$, it follows that

$$\ln 2 \approx 0.693.$$

If we assume $\beta = -\frac{2}{3}$ in the approximate formula for $\ln(1+\beta)$, we can use the same method to calculate $\ln\frac{1}{3}$ and, consequently, $\ln 3 = -\ln\frac{1}{3}$ as well. In general, if we assume $\beta = -\frac{k}{k+1}$ we obtain $\ln\left(1 - \frac{k}{k+1}\right) = \ln\frac{1}{k+1}$ and, consequently, $\ln(k+1) = -\ln\frac{1}{k+1}$. However, this method of computing logarithms is still very cumbersome. For instance, if we intend to calculate $\ln 11$, then taking $k+1 = 11$, i. e. $k = 10$, we should have $\beta = -\frac{10}{11}$, in which case the error in the approximate formula will be less than

$$\left(\frac{10}{11}\right)^{2n+1} : (2n+1)\left(1 - \frac{10}{11}\right) = \frac{11}{2n+1}\left(\frac{10}{11}\right)^{2n+1}$$

We have:

$$\frac{10}{11} \approx 0.91; \ \left(\frac{10}{11}\right)^2 \approx 0.83; \ \left(\frac{10}{11}\right)^4 \approx 0.69; \ \left(\frac{10}{11}\right)^8 \approx 0.48;$$

$$\left(\frac{10}{11}\right)^{16} \approx 0.29; \ \left(\frac{10}{11}\right)^{32} \approx 0.08; \ \left(\frac{10}{11}\right)^{64} \approx 0.006; \ \left(\frac{10}{11}\right)^{65} \approx 0.005.$$

So we see that only by taking $2n+1 = 65$ can we guarantee that the error in the approximate formula for computing $\ln\frac{1}{11}$ will be less than $\frac{11}{65} \cdot 0.005 \approx 0.001$.

It is evident that the computation of $\ln\frac{1}{11}$ in this case will be tedious, since we have to calculate the sum of 64 terms:

$$-\frac{10}{11} - \frac{1}{2}\left(\frac{10}{11}\right)^2 - \frac{1}{3}\left(\frac{10}{11}\right)^3 - \ldots - \frac{1}{64}\left(\frac{10}{11}\right)^{64}.$$

17. The conclusions we have come to concerning the approximate formula for $\ln(1+\beta)$ force us to look for another formula requiring fewer numerical calculations. Such a formula does exist, and to obtain it we take an arbitrary positive integer k and put $\beta = \frac{1}{2k+1}$. We then have

$$\ln\left(1+\frac{1}{2k+1}\right) \approx \frac{1}{2k+1} - \frac{1}{2(2k+1)^2} + \frac{1}{3(2k+1)^3} - $$
$$- \frac{1}{4(2k+1)^4} + \ldots + \frac{1}{(2n-1)(2k+1)^{2n-1}} - \frac{1}{2n(2k+1)^{2n}}.$$

The error in this approximate equality is less than $\frac{1}{(2n+1)(2k+1)^{2n+1}}$. Now let us take a negative β equal to $-\frac{1}{2k+1}$. This time we obtain another approximate equality

$$\ln\left(1-\frac{1}{2k+1}\right) \approx -\frac{1}{2k+1} - \frac{1}{2(2k+1)^2} - \frac{1}{3(2k+1)^3} - $$
$$- \frac{1}{4(2k+1)^4} - \ldots - \frac{1}{(2n-1)(2k+1)^{2n-1}} - \frac{1}{2n(2k+1)^{2n}},$$

yielding $\ln\left(1-\frac{1}{2k+1}\right)$ with an error less than

$$\frac{1}{(2k+1)^{2n+1}} : \left[(2n+1)\left(1-\frac{1}{2k+1}\right)\right] =$$
$$= \frac{2k+1}{2k} \cdot \frac{1}{2n+1} \cdot \frac{1}{(2k+1)^{2n+1}}.$$

Subtracting the second approximate equality term by term from the first we obtain

$$\ln\left(1+\frac{1}{2k+1}\right) - \ln\left(1-\frac{1}{2k+1}\right) \approx$$
$$\approx \frac{2}{2k+1} + \frac{2}{3(2k+1)^3} + \frac{2}{5(2k+1)^5} + \ldots$$
$$\ldots + \frac{2}{(2n-1)(2k+1)^{2n-1}}$$

The error in this approximate equality does not exceed, in absolute value, the sum of the errors of the formulas for $\ln\left(1+\frac{1}{2k+1}\right)$ and $\ln\left(1-\frac{1}{2k+1}\right)$, and so it is less than

$$\frac{1}{2n+1}\cdot\frac{1}{(2k+1)^{2n+1}}+\frac{2k+1}{2k}\cdot\frac{1}{2n+1}\cdot\frac{1}{(2k+1)^{2n+1}}=$$

$$=\frac{4k+1}{2k}\cdot\frac{1}{2n+1}\cdot\frac{1}{(2k+1)^{2n+1}}<$$

$$<\frac{4k+2}{2k}\cdot\frac{1}{2n+1}\,\frac{1}{(2k+1)^{2n+1}}=\frac{1}{k(2n+1)(2k+1)^{2n}}.$$

We transform the difference between the logarithms, noting that it must be equal to the logarithm of the quotient. We receive

$$\ln\left(1+\frac{1}{2k+1}\right)-\ln\left(1-\frac{1}{2k+1}\right)=\ln\frac{1+\frac{1}{2k+1}}{1-\frac{1}{2k+1}}=$$

$$=\ln\frac{2k+2}{2k}=\ln\frac{k+1}{k}=\ln(k+1)-\ln k.$$

And so we have

$$\ln(k+1)-\ln k\approx\frac{2}{2k+1}+\frac{2}{3(2k+1)^3}+\frac{2}{5(2k+1)^5}+\dots$$

$$\dots+\frac{2}{(2n-1)(2k+1)^{2n-1}}\qquad(*)$$

with an error less than $\frac{1}{k}\cdot\frac{1}{2n+1}\cdot\frac{1}{(2k+1)^{2n}}$.

This is precisely the formula we need. It allows $\ln(k+1)$ to be calculated when $\ln k$ is known. Making use of the fact that $\ln 1=0$ and assuming $k=1$, we can find $\ln 2$ with an error less than

$$\frac{1}{2n+1}\,\frac{1}{3^{2n}}.$$

Let us take $n=5$; in this case we can be sure that the error will be less than $1/11\cdot 1/3^{10}=1/(11\cdot 59049)<0.000002$.

This yields

$$\ln 2 = \ln 2 - \ln 1 \approx \frac{2}{3} + \frac{2}{3 \cdot 3^3} + \frac{2}{5 \cdot 3^5} + \frac{2}{7 \cdot 3^7} + \frac{2}{9 \cdot 3^9}$$

with an error less than 0.000002. Converting each of the five fractions into a decimal to six places (i. e. accurate to within 0.0000005) and adding we obtain the value of $\ln 2$ with an accuracy to within $0.000002 + 0.0000005 \cdot 5 < 0.000005$:

$$\ln 2 \approx 0.693146 \approx 0.69315.$$

Now putting $k = 2$ and $n = 3$ in formula $(*)$ we receive

$$\ln 3 - \ln 2 \approx \frac{2}{5} + \frac{2}{3 \cdot 5^3} + \frac{2}{5 \cdot 5^5} \approx 0.40546$$

with an error less than $1/2 \cdot 1/7 \cdot 1/5^6 = \dfrac{1}{14 \cdot 15625} < 0.000005.$

Therefore

$$\ln 3 \approx \ln 2 + 0.40546 \approx 1.09861.$$

Next we obtain $\ln 4 = 2 \ln 2 \approx 1.38630$; assuming $k = 4$ and $n = 3$ in formula $(*)$ we receive

$$\ln 5 - \ln 4 \approx \frac{2}{9} + \frac{2}{3 \cdot 9^3} + \frac{2}{5 \cdot 9^5} \approx 0.223144 \approx 0.22314$$

with an error less than

$$\frac{1}{4} \cdot \frac{1}{7} \cdot \frac{1}{9^6} = \frac{1}{28 \cdot 531441} < 0.0000001$$

and consequently

$$\ln 5 \approx \ln 4 + 0.22314 \approx 1.60944.$$

Now we can easily find $\ln 10$:

$$\ln 10 = \ln 5 + \ln 2 \approx 2.30259.$$

And, finally, putting $k = 10$ and $n = 2$ in formula $(*)$ we obtain

$$\ln 11 - \ln 10 \approx \frac{2}{21} + \frac{3}{3 \cdot 21^3} \approx 0.09531$$

(here the error in the approximate formula is less than $1/10 \cdot 1/5 \cdot 1/21^4 \approx 0.0000001$).
Therefore

$$\ln 11 \approx \ln 10 + 0.09531 \approx 2.39790.$$

These examples are sufficient to understand how the table of natural logarithms is constructed. This is the way to construct the following table of logarithms of integers from 1 to 100, calculated with an accuracy to within the factor 0.0005.

Table of Natural Logarithms (from 1 to 100)

n	$\ln n$	n	$\ln n$	n	$\ln n$	n	$\ln n$	n	$\ln n$
1	0.000	21	3.045	41	3.714	61	4.111	81	4.394
2	0.693	22	3.091	42	3.738	62	4.127	82	4.407
3	1.099	23	3.135	43	3.761	63	4.143	83	4.419
4	1.386	24	3.178	44	3.784	64	4.159	84	4.431
5	1.609	25	3.219	45	3.807	65	4.174	85	4.443
6	1.792	26	3.258	46	3.829	66	4.190	86	4.454
7	1.946	27	3.296	47	3.850	67	4.205	87	4.466
8	2.079	28	3.332	48	3.871	68	4.220	88	4.477
9	2.197	29	3.367	49	3.892	69	4.234	89	4.489
10	2.303	30	3.401	50	3.912	70	4.248	90	4.500
11	2.398	31	3.434	51	3.932	71	4.263	91	4.511
12	2.485	32	3.466	52	3.951	72	4.277	92	4.522
13	2.565	33	3.497	53	3.970	73	4.290	93	4.533
14	2.639	34	3.526	54	3.989	74	4.304	94	4.543
15	2.708	35	3.555	55	4.007	75	4.317	95	4.554
16	2.773	36	3.584	56	4.025	76	4.331	96	4.564
17	2.833	37	3.611	57	4.043	77	4.344	97	4.575
18	2.890	38	3.638	58	4.060	78	4.357	98	4.585
19	2.944	39	3.664	59	4.078	79	4.369	99	4.595
20	2.996	40	3.689	60	4.094	80	4.382	100	4.605

18. We have seen that the logarithm of a product can be found by means of addition, the logarithm of a quotient by means of subtraction, the logarithm of a power by multiplication (by the exponent), and the logarithm of a root by division (by the index of the root). Therefore, if we have a table in which logarithms are written next to the numbers (a table of logarithms), then with the aid of that table we can replace multiplication by addition, division by subtraction, raising to a power by multiplication and obtaining a root by division, that is to say, every time a more complex operation is replaced by a simpler one. You can learn how to do it from a high school algebra book so here we shall only consider a simple example.

Say we want to calculate $c = \sqrt[5]{2}$. Making use of the value $\ln 2 \approx 0.693$ computed above let us divide it by 5 to obtain $\ln \sqrt[5]{2} = 1/5 \ln 2 \approx 0.139$. It remains to find the number $\sqrt[5]{2}$ by its logarithm. Our table is not very convenient, for it contains logarithms 0.000 (corresponding to the number 1) and 0.693 (corresponding to the number 2), the first being too small and the second too large. From this we can only conclude that $1 < \sqrt[5]{2} < 2$. But it can be noted that $\ln(10\sqrt[5]{2}) = \ln 10 + \ln \sqrt[5]{2} = 2.303 + 0.139 = 2.442$. In our table the nearest smaller logarithm is 2.398 ($= \ln 11$), and the nearest larger logarithm is 2.485 ($= \ln 12$), consequently $11 < 10\sqrt[5]{2} < 12$. Noting that 2.442 lies approximately between $\ln 11$ and $\ln 12$ (the arithmetic mean of the last numbers is 2.441) we can put $10\sqrt[5]{2} \approx 11.5$, i. e. $\sqrt[5]{2} \approx 1.15$. To check the result obtained take note of the fact that

$$\ln(100\sqrt[5]{2}) = \ln 100 + \ln \sqrt[5]{2} = 4.605 + 0.139 = 4.744$$

and

$$\ln 115 = \ln 5 + \ln 23 = 1.609 + 3.135 = 4.744.$$

19. To construct the graph of the function $y = \ln x$, it is necessary to choose coordinate axes and a scale unit and, then, for every x ($x > 0$) mark off the value of $\ln x$ on the line perpendicular to the x-axis and raised from the respective point on that axis. The end points of the perpendiculars obtained for various values of x will be located on a curve constituting the graph of the natural logarithm. The graph of the logarithm is shown in Fig. 30*a*. Fig. 30*b*, located below, depicts $\ln x$ as an area, so that the two cases can be compared. Both figures are drawn to the same scale. If we take one and the same value of x, we can show that the number of unit squares contained in the area of the curvilinear trapezoid $ACDB$ in Fig. 30*b* is equal to the number of units of length contained in line segment KL shown in Fig. 30*a*.

Let us note that if $0 < x' < 1$, then

$$\ln x' = \int_1^{x'} \frac{dx}{x} = -\int_{x'}^{1} \frac{dx}{x},$$

that is, $\ln x'$ is a negative number whose absolute value is equal to the area of the trapezoid $B'D'CA$; therefore, for this case $\ln x'$

will be shown in Fig. 30*a* by a line segment $K'L'$ marked off downwards from the x-axis.

All the properties of the graph of the function $y = \ln x$ follow from the definition and the properties of the natural logarithm. For example, $\ln x$ is negative for $x < 1$, vanishes for $x = 1$ and is positive for $x > 1$. Consequently, the graph of the logarithm is located below Ox for $x < 1$, intersects Ox for $x = 1$ and is above Ox for $x > 1$. Further, $y = \ln x$ increases with increasing x.

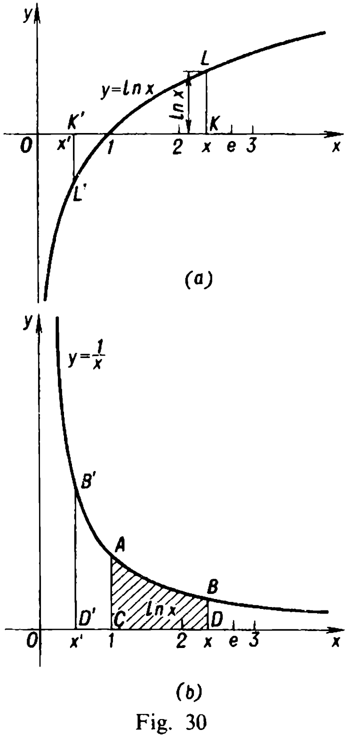

Fig. 30

This property is obvious when $x > 1$ (see Fig. 30*b*), but it is also true for $x = x' < 1$. Indeed, if x' increases, remaining less than unity, the absolute value of the area $B'D'CA$ (Fig. 30*b*) decreases, and this means that $\ln x$, which differs from that area only in sign, increases.

This increase in the logarithm is seen in the graph in the form of a curve rising as if uphill from left to right. This

hill, steep at first, becomes flatter and flatter. For greater clarity, we shall call the graph of the logarithm depicted here a logarithmic slope.

If we cut a horizontal path along the x-axis and proceed along this path from the point O to the right, then, looking down, we first see an infinite gulf with the logarithmic slope lost in its depth. However, we need only to take a step equal to the unit of length to leave the gulf behind. Continuing our walk along the path, with each step we find ourselves higher and higher on the slope. Thus, after two steps ($x = 2$) our height will be $\ln 2 = 0.693$, after three steps $\ln 3 = 1.099$, etc. Let us calculate how great the increase in the height of the slope will be when after m steps we take one more step. Since after m steps (a unit of length each) the height of the slope will be equal to $\ln m$, and after $m + 1$ steps it will be $\ln(m + 1)$, the increase in the height of the slope corresponding to one step is

$$\ln(m + 1) - \ln m = \ln \frac{m + 1}{m} = \ln\left(1 + \frac{1}{m}\right).$$

The greater the number of steps we take the less will be the number $\frac{1}{m}$, the more $1 + \frac{1}{m}$ will approach unity, and the closer $\ln\left(1 + \frac{1}{m}\right)$ will be to zero. This means that the steepness of the slope becomes less and less noticeable as we move to the right, i.e. the logarithmic slope becomes indeed less steep.

The comparative flatness of the slope does not prevent it from rising indefinitely, so that when we go sufficiently far along the horizontal path the slope will rise above us indefinitely. Indeed, after we make 2^m steps the height of the slope will be equal to

$$\ln 2^m = m \ln 2 = 0.693\, m,$$

and for a sufficiently large m this number will be arbitrarily large.

If instead of a horizontal path we cut some other straight path with a rise, however slight (Fig. 31a), and travel along it, then, sooner or later we shall reach the logarithmic slope, and shall even leave it down below us when we rise further upwards (Fig. 31b).

To be sure of this let us prove the following lemma: for any

natural m there holds the inequality

$$\frac{4^m}{m^2} \geqslant 4.$$

Indeed, when m increases by 1 the fraction $\frac{4^m}{m^2}$ increases, i. e.

$$\frac{4^m}{m^2} < \frac{4^{m+1}}{(m+1)^2};$$

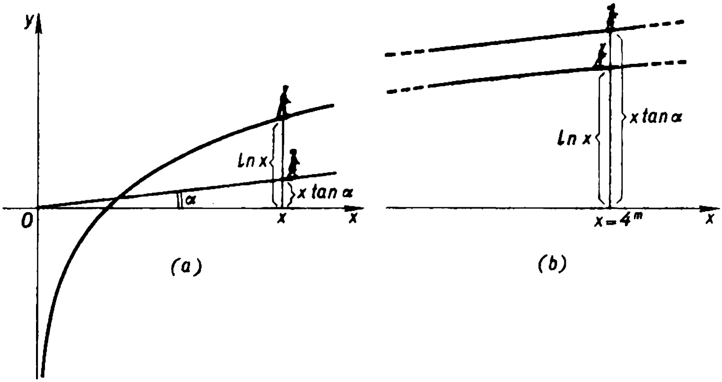

Fig. 31

this follows from the inequality

$$\frac{4^{m+1}}{(m+1)^2} : \frac{4^m}{m^2} = \frac{4m^2}{(m+1)^2} = \left(\frac{2m}{m+1}\right)^2 = \left(\frac{m+m}{m+1}\right)^2 \geqslant 1,$$

which holds for $m \geqslant 1$. This is why from among the fractions

$$\frac{4^1}{1^2}, \frac{4^2}{2^2}, \ldots, \frac{4^m}{m^2}, \ldots$$

the first always has the lowest value, i. e.

$$\frac{4^1}{1^2} \leqslant \frac{4^m}{m^2};$$

this is what we wanted to prove.

Now note that for each point on the sloping straight path there holds the relation

$$y = x \tan \alpha,$$

where α is the angle of inclination of the path (α is an acute angle and, consequently, $\tan\alpha > 0$). If we take $x = 4^m$, the altitude of the slope of the path for this value of x will be $4^m \tan\alpha$, and the height of the logarithmic slope will equal $\ln(4^m) = m \ln 4$. The ratio of the first height to the second will be

$$\frac{4^m \tan\alpha}{m \ln 4} = \frac{4^m \tan\alpha}{m^2 \ln 4} m.$$

But according to what was proved above, $\frac{4^m}{m^2} \geqslant 4$, and therefore the ratio between the height of the path and the height of the logarithmic slope is not less than $\frac{4\tan\alpha}{\ln 4} m$, and for a sufficiently large m this value can be made arbitrarily great. Consequently, for $x = 4^m$ and a high value of m the rising straight path will be considerably higher than the logarithmic slope (see Fig. 31*b*).

It is remarkable that the logarithmic slope has a rounded shape, without any irregularities, and is convex throughout. This property can be expressed in geometric terms: every arc of the graph of the logarithm lies above the chord of that arc (Fig. 32). Denoting the abscissas of the end points of an arbitrary arc

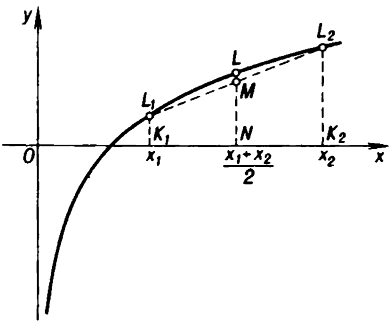

Fig. 32

L_1L_2 by x_1x_2 we can make sure that for the mean value of $x = \frac{x_1 + x_2}{2}$ a point on the arc L must indeed lie above the corresponding midpoint of the chord M.

In fact,

$$NL = \ln \frac{x_1 + x_2}{2}$$

and

$$NM = \frac{K_1L_1 + K_2L_2}{2}$$

(as the middle line of the trapezoid), i. e.

$$NM = \frac{\ln x_1 + \ln x_2}{2}.$$

We have to prove that

$$\ln \frac{x_1 + x_2}{2} > \frac{\ln x_1 + \ln x_2}{2}.$$

But we have

$$\frac{\ln x_1 + \ln x_2}{2} = \frac{1}{2} \ln (x_1 x_2) = \ln \sqrt{x_1 x_2}.$$

Therefore it is necessary to prove that

$$\ln \frac{x_1 + x_2}{2} > \ln \sqrt{x_1 x_2}.$$

Let us note that

$$(\sqrt{x_1} - \sqrt{x_2})^2 = x_1 - 2\sqrt{x_1 x_2} + x_2 > 0$$

(in case x_1 and x_2 are positive numbers not equal to each other). Therefore

$$x_1 + x_2 > 2\sqrt{x_1 x_2},$$

next

$$\frac{x_1 + x_2}{2} > \sqrt{x_1 x_2}$$

and finally

$$\ln \frac{x_1 + x_2}{2} > \ln \sqrt{x_1 x_2}.$$

The proof is complete.

Thus, whatever the arc of the graph of the logarithm, the point of the arc corresponding to the arithmetic mean of the abscissas of its end points always lies above the midpoint of the chord. This implies that the graph of a logarithm can have no troughs.

Indeed, if there were such a trough (Fig. 33), an arc would be possible for which the above property would not hold (the midpoint of the chord M would lie not below, but above the corresponding point of the arc L).

Proceeding from the properties of logarithms, we could derive some other interesting properties of the graph of a logarithm, but we shall restrict ourselves to those just considered.

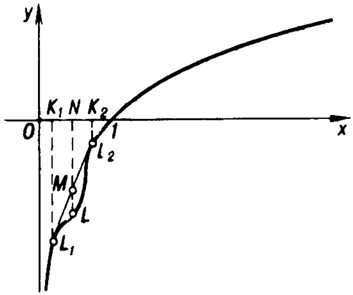

Fig. 33

20. We may encounter natural logarithms while solving many problems in mathematics and physics which at first glance appear to have nothing to do with the areas of curvilinear trapezoids bounded by the arcs of a hyperbola. Here is one of the problems of this kind studied by the eminent Russian mathematician P. L. Chebyshev: he wanted to find the simplest formula possible for an approximate calculation of all prime numbers not exceeding some given (arbitrary) number n.

If n is not large, then the question as to the quantity of prime numbers denoted by $\pi(n)$ (here π has nothing in common with the familiar number 3.14159...) is decided very simply. Thus, if $n = 10$, the prime numbers not exceeding 10 are the following: 2, 3, 5, 7; there are four of them and consequently $\pi(10) = 4$. If $n = 100$, then we make use of the familiar method known as Eratosthenes' sieve and obtain 25 prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97; consequently $\pi(100) = 25$. However for a large n the problem becomes rather difficult. How can we calculate $\pi(n)$, even very approximately, when n is equal to a million, a thousand million, and so on?

Chebyshev found that to calculate $\pi(n)$ approximately it is

sufficient to divide n by the natural logarithm of n:

$$\pi(n) \approx \frac{n}{\ln n};$$

the relative error of this equality (the error measured in fractions of the number $\pi(n)$) may be very large, but it tends to zero when n tends to infinity. Chebyshev's approximation formula becomes very convenient in the case of n equal to a power of 10 with a positive integral exponent: $n = 10^k$. Then we obtain $\ln n = \ln 10^k = k \ln 10 \approx$ $\approx 2.303\,k$ and, hence,

$$\pi(10^k) \approx \frac{10^k}{2.303\,k}.$$

Making use of the fact that $\frac{1}{2.303} \approx 0.434$, we can obtain a formula still more convenient for computations:

$$\pi(10^k) \approx 0.434\frac{10^k}{k}.$$

Thus, for $k = 1$ and $k = 2$ we find:

$$\pi(10) \approx 0.434 \cdot 10 = 4.34 \text{ (the correct result is 4)},$$

$$\pi(100) \approx 0.434 \cdot \frac{100}{2} \approx 21.7 \text{ (the correct result is 25)}.$$

If we continued the calculations we would obtain the following:

$$\pi(1000) \approx 0.434 \cdot \frac{1000}{3} \approx 145 \text{ (the correct result is 168)},$$

$$\pi(10\,000) \approx 0.434 \cdot \frac{10\,000}{4} \approx 1090 \text{ (the correct result is 1229)},$$

$$\pi(10^6) \approx 0.434 \cdot \frac{10^6}{6} \approx 72\,300 \text{ (the correct result is 78\,498)}.$$

The relative error of the last result is

$$\frac{78\,498 - 72\,300}{78\,498} \approx 0.08,$$

i. e. 8 per cent, it is still considerable. However, a very rigorous proof can be presented that the relative error of Chebyshev's formula can be made infinitely small if 10^k is sufficiently large. At some stage it will be smaller than one per cent. Further it

will become less than 0.1 per cent; next, less than 0.001 per cent, etc. This explains why Chebyshev's formula is of great theoretical value.

To Chebyshev's credit is another formula for an approximate calculation of $\pi(n)$, somewhat more difficult but allowing a better approximation:

$$\pi(n) \approx \int_2^n \frac{dt}{\ln t}.$$

Without doing the calculations we shall give here only some results:

$$\int_2^{1000} \frac{dt}{\ln t} \approx 177 \qquad (\pi(1000) = 168);$$

$$\int_2^{10\,000} \frac{dt}{\ln t} \approx 1245 \qquad (\pi(10\,000) = 1229),$$

$$\int_2^{1\,000\,000} \frac{dt}{\ln t} \approx 78\,627 \qquad (\pi(1\,000\,000) = 78\,498).$$

Hence, the relative error of the approximate equality

$$\pi(1\,000\,000) \approx \int_2^{1\,000\,000} \frac{dt}{\ln t}$$

is equal to

$$\frac{|78\,498 - 78\,627|}{78\,498} \approx 0.0016,$$

that is 0.16 per cent.

21. We saw that

$$\ln 2 = 0.69315 < 1, \text{ and } \ln 3 = 1.09861 > 1.$$

This means that the area of $ACDB$ (Fig. 34) is less than 1, and the area of ACD_1B_1 is greater than 1. It can be expected that somewhere between the points D and D_1 there will be found a point D' such that the area of $ACD'B'$ will be equal to 1. Such a point D' does exist. If we denote OD' by e we may state that $2 < e < 3$. Making use of the table of logarithms given on

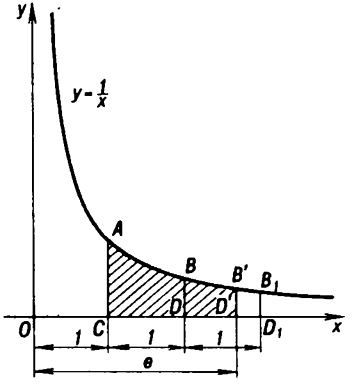

Fig. 34

p. 45 we find that $2.7 < e < 2.8$. Indeed,

$$\ln 2.7 = \ln 27 - \ln 10 \approx 0.993$$

and

$$\ln 2.8 = \ln 28 - \ln 10 = 1.029.$$

There exist various methods of finding e with any degree of accuracy. Without considering them in detail, we shall only give the result:

$$e \approx 2.71828$$

(all the digits written here are correct). By definition

$$\ln e = 1.$$

The number e is termed the *base* of natural or Napierian logarithms, named after John Napier, a Scottish mathematician who published the first table of logarithms (in 1614).

Proceeding from the properties of natural logarithms we can prove the following remarkable proposition: the natural logarithm of any positive number b is equal to the exponent to which the number e should be raised in order to obtain b. In other words, if $\ln b = \alpha$, then $b = e^{\alpha}$. For example, from the fact that $\ln 2 \approx 0.69315$

it follows that $2 \approx e^{0.69315}$; from $\ln 10 \approx 2.30259$ it follows that $10 \approx e^{2.30259}$, etc.

To prove this it is sufficient to use the property of the logarithm of a power. Let $b = e^x$; then $\ln b = \ln e^x = x \ln e$. But $\ln e = 1$, therefore

$$\ln b = x,$$

i. e. the natural logarithm of b coincides with the exponent x.

Thus we see that natural logarithms can be determined without resort to geometric representations. It could be said from the very beginning that the natural logarithm of the number b is an exponent of the power to which the number $e \approx 2.71828$ must be raised to obtain the number b. But this definition does not show clearly enough why we are interested in the exponents of the power of the number e and not of some other number. Now if natural logarithms are introduced as areas, their definition becomes pictorial enough, and does not leave any doubt.

We should say, at this point, that besides natural logarithms, some other logarithms can be introduced, with another base. Thus, for instance, a common logarithm of the number b is the exponent of the power to which the number 10 should be raised to obtain the number b. The common logarithm of the number b is denoted $\log b$. If we have $\log b = \beta$, then, by definition, we must have $b = 10^\beta$; it is evident that $\log 10 = 1$. Common logarithms are studied in high school, and there all their properties are derived not by geometrical means, but on the basis of the known properties of power exponents.

There is a simple relationship between common and natural logarithms. Let $\ln b = \alpha$ and $\log b = \beta$. This means that $b = e^\alpha$ and $b = 10^\beta$, i. e. $e^\alpha = 10^\beta$. Consequently, $\ln e^\alpha = \ln 10^\beta$ or $\alpha \ln e = \beta \ln 10$, i. e. $\alpha = \beta \cdot 2.30259$. Thus we have $\ln b = 2.30259 \log b$, whence

$$\log b = \frac{1}{2.30259} \ln b = 0.43429 \ln b.$$

Having before us the table of natural logarithms and multiplying each logarithm by 0.43429 we obtain the table of common logarithms.

For example,

$$\log 2 = 0.43429 \ln 2 = 0.43429 \cdot 0.69315 \approx 0.30103.$$

For $\log 10$ we must obtain unity:

$$\log 10 = 0.43429 \ln 10 \approx 0.43429 \cdot 2.30259 = 1.$$

The fact that the number 10 is taken as the base of common logarithms (the number 10 is the base of the decimal system of notation) considerably simplifies logarithmic computations. Thus, knowing that $\log 2 = 0.30103$ and $\log 10 = 1$, we immediately obtain

$$\log 20 = \log 2 + \log 10 = \log 2 + 1 = 1.30103,$$

$$\log 200 = \log 2 + \log 100 = \log 2 + 2 = 2.30103,$$

and so on.

Now, if we know that $\ln 2 = 0.69315$ and $\ln 10 = 2.302585$ and want to calculate $\ln 20$ and $\ln 200$, we should proceed as follows:

$$\ln 20 = \ln 2 + \ln 10 = 0.69315 + 2.30259 = 2.99574,$$

$$\ln 200 = \ln 2 + \ln 100 = \ln 2 + 2 \ln 10 = \\ = 0.69315 + 4.60517 = 5.29832.$$

This explains why when using logarithms as an auxiliary means in computations we prefer to employ the tables of common logarithms. But this in no way belittles the importance of natural logarithms, which are encountered in the solution of many problems in mathematics and the natural sciences. In this book we discussed two mathematical problems leading to natural logarithms, that is the problem concerning the area of an equilateral hyperbola and Chebyshev's problem on the distribution of prime numbers.

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1. To calculate the area of a curvilinear trapezoid we replaced it either by the sum of the areas of rectangles (p. 8), or by the sum of the areas of rectilinear trapezoids (p. 33). Using the latter method we got for $\ln 2$ the approximate value 0.6937. On p. 44 we employed another method to compute $\ln 2$ and obtained the value 0.69315, which contains an error less than 0.000005. Hence we see that the calculation with the aid of trapezoids led to an error greater than 0.0005, notwithstanding the fact that we divided the line segment CD in Fig. 26*a* into many parts (10) so that the altitudes of the rectilinear trapezoid represented by these segments constituted only 0.1 each.

There are other methods of approximation suitable for any curvilinear trapezoid and leading to highly accurate results while requiring computations no more difficult than with the methods just described. We shall now consider a method named after an eminent English mathematician Thomas Simpson (1710-1761), although a method like it was suggested 75 years earlier by his countryman James Gregory (1638-1675). The main idea is to replace the arcs of the graph of the function, not by chords, as was done in the case of rectilinear trapezoids (see Fig. 26*b*), but by the arcs of *parabolas*.

In high-school algebra books the term parabola is used to describe the graph of the function $y = ax^2$. Here we shall use this term in a wider sense, describing as a parabola the graph of any function of the form $y = ax^2 + bx + c$. For $a \neq 0$ and $b = c = 0$ it is a parabola in its ordinary position, with a vertex at the origin. If $a \neq 0$, as before, but one of the coefficients, b or c, is also different from zero, then, as can be verified, the graph of the function is the same parabola but with its vertex transferred to some point different from the origin (Fig. 35). Finally, if $a = 0$, we have a straight line $y = bx + c$. Taking this into account, we shall continue to call it here a parabola, interpreting it as a special case.

Let us prove now that through three points of a plane, $A(x_0, y_0)$, $B(x_1, y_1)$, $C(x_2, y_2)$, with pairwise distinct abscissas x_0, x_1 and x_2 we can draw one and only one parabola. This means that there exist coefficients a, b and c such that the graph of the function $y = ax^2 + bx + c$ will pass through each of the indicated points. In this case the values of the coefficients are determined uniquely by the choice of the points A, B and C.

To prove this assertion we take a, b and c as unknowns.

The coefficients a, b and c we have to find must satisfy the following three conditions: the graph of the function $y = ax^2 + bx + c$ must pass (1) through the point $A(x_0, y_0)$, (2) through the point $B(x_1, y_1)$ and (3) through the point $C(x_2, y_2)$. In other words, the value of the required function is y_0 for $x = x_0$, y_1 for $x = x_1$ and $y = y_2$ for $x = x_2$. Therefore we obtain three equations

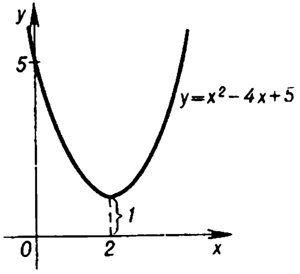

Fig. 35

with three unknowns a, b and c:

$$y_0 = ax_0^2 + bx_0 + c, \tag{1}$$

$$y_1 = ax_1^2 + bx_1 + c, \tag{2}$$

$$y_2 = ax_2^2 + bx_2 + c. \tag{3}$$

If we subtract termwise the first equation from the second and the second from the third, we shall have a system of two equations with the unknowns a and b:

$$y_1 - y_0 = a(x_1^2 - x_0^2) + b(x_1 - x_0), \tag{1'}$$

$$y_2 - y_1 = a(x_2^2 - x_1^2) + b(x_2 - x_1). \tag{2'}$$

Here it is convenient to divide all the terms of the first equation by $x_1 - x_0$, and of the second by $x_2 - x_1$. By the hypothesis these numbers are known and are different from zero. So we obtain

$$\frac{y_1 - y_0}{x_1 - x_0} = a(x_1 + x_0) + b, \tag{1''}$$

$$\frac{y_2 - y_1}{x_2 - x_1} = a(x_2 + x_1) + b. \tag{2''}$$

Now we subtract the first equation of the last system from the second equation and obtain a single equation defining a:

$$\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0} = a(x_2 - x_0).$$

By dividing all the terms of the last equation by the number $x_2 - x_0$ (not equal to zero) we find the only possible value of the required coefficient a. Let us substitute it, say, into equation (1″). Then from this equation we immediately find the only possible value of coefficient b. Finally, substituting the values of a and b thus found into equation (1) we find the only possible value of the last unknown coefficient c. We believe the presentation of the calculations here to be unnecessary. The expressions for a, b and c will evidently be rational numbers, specified by the coordinates of the three given points (verify this, beginning with a).

Thus it follows from our discussion that the coefficients a, b and c cannot possess any other values except those given above. Hence there exists only one parabola passing through the three given points. It is easy to verify that the obtained values of a, b and c satisfy equations (1), (2) and (3).

2. Now take three points $A(x_0, y_0)$, $B(x_1, y_1)$ and $C(x_2, y_2)$ with pairwise distinct abscissas such that $x_0 < x_1 < x_2$ and x_1 is located exactly in the middle of the segment $[x_0, x_2]$. This means that $x_1 = \frac{x_0 + x_2}{2}$. According to what was proved above one and only one parabola, $y = ax^2 + bx + c$, passes through them (Fig. 36). Let us consider the area S of the curvilinear

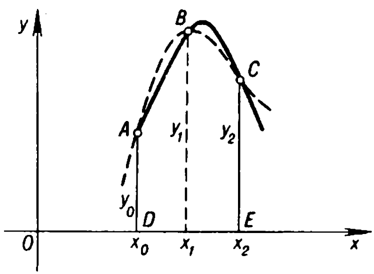

Fig. 36

trapezoid $ADEC$. We shall prove that it is equal to $\frac{x_2 - x_0}{6}(y_0 + 4y_1 + y_2)$. In other words, for the parabola passing through the given points the following formula is valid:

$$S = \frac{x_2 - x_0}{6}(y_0 + 4y_1 + y_2). \tag{4}$$

The case is not excluded when all the three points lie on a single straight line. If this line is parallel to the x-axis (Fig. 37a), then $y_2 = y_1 = y_0$, and from formula (4) we obtain the expression for S: $S = \frac{x_2 - x_0}{6} \cdot 6y_0 = (x_2 - x_0)y_0$. But this is precisely the area of the corresponding rectangle (according to the condition introduced on p. 7 it is expressed by a negative number if $y_0 < 0$). Now if the straight line is not parallel to the x-axis (Fig. 37b), then S is equal to the area of the rectilinear

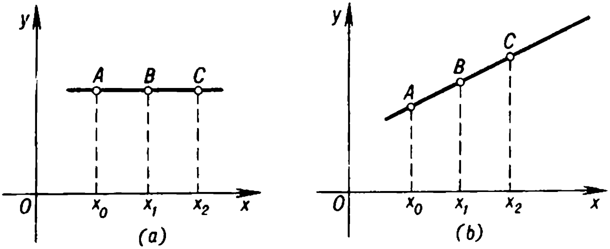

Fig. 37

trapezoid taken with the requisite sign. Its midline is $y_1 = \frac{y_0 + y_2}{2}$ and its altitude is equal to $x_2 - x_0$. Substituting in formula (4) $2y_1$ for the sum $y_0 + y_2$, we receive $S = \frac{x_2 - x_0}{6}(2y_1 + 4y_1) = (x_2 - x_0)y_1$, again a correct result. By the way, to prove formula (4) there is no need to specially consider each individual case. In all cases the proof is the same.

Let us calculate the coefficients a, b and c by the technique considered in the previous section. Then $y = ax^2 + bx + c$ is a function whose graph passes through the given points. This means that the three equalities (1), (2) and (3) are satisfied. Now we shall prove formula (4) without expressing the coefficients a, b and c in terms of the coordinates of the points A, B and C (in fact, we did not give these calculations in Section 1). We shall simply make sure that the formula is valid. The reader is asked to accept on trust that the reasoning of the proof is correct; otherwise some cumbersome computations would be necessary to find the expressions for a, b and c.

First we express the area S, which we can term the area of a parabolic trapezoid, as an integral. Using the known properties of

integrals and the formulas for $\int_{x_0}^{x_2} x^2 dx$, $\int_{x_0}^{x_2} x\,dx$ and $\int_{x_0}^{x} dx$, we obtain

$$S = \int_{x_0}^{x_2} (ax^2 + bx + c)dx = a\int_{x_0}^{x_2} x^2\,dx + b\int_{x_0}^{x_2} x\,dx + c\int_{x_0}^{x_2} dx =$$

$$= a\,\frac{x_2^3 - x_0^3}{3} + b\,\frac{x_2^2 - x_0^2}{2} + c(x_2 - x_0) =$$

$$= \frac{2a(x_2^3 - x_0^3) + 3b(x_2^2 - x_0^2) + 6c(x_2 - x_0)}{6}.$$

All the binomials placed in parentheses here have a common factor $x_2 - x_0$. This is obvious for the last binomial and, besides, $x_2^2 - x_0^2 = (x_2 - x_0)(x_2 + x_0)$ and $x_2^3 - x_0^3 = (x_2 - x_0)(x_2^2 + x_2x_0 + x_0^2)$. Consequently, moving the common factor outside the parentheses we express S in the form

$$S = \frac{x_2 - x_0}{6}[a(2x_2^2 + 2x_2x_0 + 2x_0^2) + b(3x_2 + 3x_0) + 6c].$$

Comparing the result with formula (4), which we are trying to prove, we see that it only remains to verify the equation

$$a(2x_2^2 + 2x_2x_0 + 2x_0^2) + b(3x_2 + 3x_0) + 6c = y_0 + 4y_1 + y_2. \tag{5}$$

Use the expressions (1), (2) and (3) for y_1, y_2 and y_3. Since the left-hand side of equation (5) does not contain the abscissa x_1 of the point B, we replace x_1 by the expression $\frac{x_0 + x_2}{2}$ before substituting the expression for y_1. We then have

$$y_1 = a\left(\frac{x_0 + x_2}{2}\right)^2 + b\left(\frac{x_0 + x_2}{2}\right) + c,$$

whence

$$4y_1 = a(x_0^2 + 2x_0x_2 + x_2^2) + b(2x_0 + 2x_2) + 4c.$$

Consequently

$$y_0 + 4y_1 + y_2 = (ax_0^2 + bx_0 + c) +$$
$$+ [a(x_0^2 + 2x_0x_2 + x_2^2) + b(2x_0 + 2x_2) + 4c] + (a_2x_2^2 + bx_2 + c) =$$
$$= a(2x_0^2 + 2x_0x_2 + 2x_2^2) + b(3x_0 + 3x_2) + 6c$$

(we have combined the terms containing the same coefficient a, b or c). Thus we see that formula (5) is valid and consequently formula (4) is valid too.

Applying this formula we can immediately see, for instance, that in the case of $x_0 = 2$, $x_1 = 3$, $x_2 = 4$, and $y_0 = 2$, $y_1 = 4$, $y_2 = 3$, the area of the parabolic trapezoid $ADEC$ (see Fig. 36) is equal to $\frac{4-2}{6}(2 + 4 \cdot 4 + 3) = 7$. This is the exact result. But a graph of some other function which is not a parabola can pass through the same points A, B and C (in Fig. 36 it is shown by a dashed line). If we replace this dashed line by a solid line, an arc of a parabola, and then calculate the area of the corresponding curvilinear trapezoid, the previous result, 7, will no longer be exact, but approximate. Making use of this idea let us once more find an approximate value of $\ln 2$. We made an approximate calculation of the integral $\int_1^2 \frac{dx}{x}$, and for this purpose make an approximate replacement of the arc of the hyperbola $y = \frac{1}{x}$ by an arc of the parabola passing through the points A, B and C (Fig. 38), for

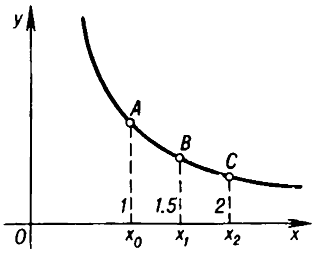

Fig. 38

which $x_0 = 1$, $x_2 = 2$, $x_1 = \frac{x_0 + x_2}{2} = 1.5$, and $y_0 = \frac{1}{x_0} = 1$, $y_2 = \frac{1}{x_2} = \frac{1}{2}$ and $y_1 = \frac{1}{x_1} = \frac{2}{3}$. The parabola is not given in the figure since in this case it differs very little from

the arc of the hyperbola. Applying formula (4) we obtain

$$\ln 2 = \int_1^2 \frac{dx}{x} \approx \frac{1}{6}\left(1 + 4\cdot\frac{2}{3} + \frac{1}{2}\right) = \frac{25}{36} = 0.694\ldots$$

The approximation we have obtained is rather close; a more exact value of $\ln 2$ is 0.69315 (see p. 44).

3. To attain an error as small as possible in computing $\int_a^b f(x)\,dx$, the interval between a and b is divided into n equal parts. Then the are of the graph of the function $y = f(x)$ is also divided into n arcs. In accordance with what was said above we replace each of them by the arc of a parabola. Then we obtain an approximate expression for the integral as the sum of the areas of n parabolic trapezoids. Each of them can be found separately with the aid of formula (4). As a result we can find an approximate expression for the integral.

All the above can be expressed as a formula named after Simpson.

Let us designate, in turn, the abscissas of the points dividing the line segment between a and b into n equal parts by the letter x with even indices: $x_0 = a$, x_2, x_4, $\ldots$, x_{2n-2}, $x_{2n} = b$ (in Fig. 39 $n = 8$). Now x with odd indices will designate the

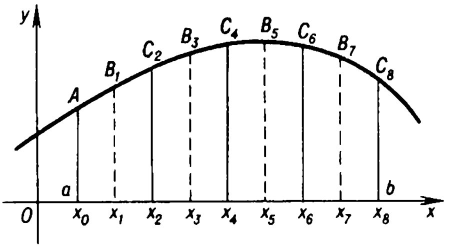

Fig. 39

midpoints of the corresponding parts, i.e. $x_1 = \dfrac{x_0 + x_2}{2}$, $x_3 = \dfrac{x_2 + x_4}{2}, \ldots, x_{2n-1} = \dfrac{x_{2n-2} + x_{2n}}{2}$. Each of the arcs of the graph AB_1C_2, $C_2B_3C_4$, $\ldots$, $C_{2n-2}B_{2n-1}C_{2n}$ is replaced by the arc of the parabola passing through three points: the end-points of the

arc and the point located above the middle of the corresponding interval of the x-axis. These are not in the drawing since they almost merge with the arcs of the graph under consideration. The areas of the parabolic trapezoids derived by formula (4) are expressed as follows:

$$\frac{x_2 - x_0}{6}(y_0 + 4y_1 + y_2),\ \frac{x_4 - x_2}{6}(y_2 + 4y_3 + y_4), \ldots$$
$$\ldots,\ \frac{x_{2n} - x_{2n-2}}{6}(y_{2n-2} + 4y_{2n-1} + y_{2n}),$$

and in accordance with the above reasoning their sum gives an approximate value of the integral $\int_a^b f(x)\,dx$. Before putting down this sum note that the difference between the two x's and the neighbouring even numbers is $\frac{b-a}{n}$:

$$x_2 - x_0 = x_4 - x_2 = \ldots = x_{2n} - x_{2n-2} = \frac{b-a}{n}.$$

Consequently, putting the common factor $\frac{b-a}{6n}$ outside the brackets we obtain:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6n}[(y_0 + 4y_1 + y_2) +$$
$$+ (y_2 + 4y_3 + y_4) + \ldots + (y_{2n-2} + 4y_{2n-1} + y_{2n})],$$

i. e. we finally obtain the expression

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6n}[(y_0 + y_{2n}) +$$
$$+ 2(y_2 + y_4 + \ldots + y_{2n-2}) + 4(y_1 + y_3 + \ldots + y_{2n-1})]. \quad (6)$$

This is Simpson's formula in its general form. The extreme ordinates in square brackets are taken with the coefficient 1, all the other ordinates with even indices are taken with the coefficient 2, and those with odd indices with the coefficient 4.

Formula (4) in the preceding article can be considered as a special case of Simpson's formula, when we use only one parabolic trapezoid, i. e. $n = 1$. We see, for instance, that it gives $\ln 2$ with an error of the order of 0.001. Let us make sure that for $n = 5$ Simpson's formula allows us to calculate $\ln 2$ with an error of the order of 0.0000001. Thus, let us make use of Simpson's formula (6) to calculate $\int\limits_1^2 \frac{dx}{x}$ taking $n = 5$. Here $a = 1$, $b = 2$, $f(x) = \frac{1}{x}$; for $n = 5$ we obtain $x_0 = 1$, $x_2 = 1.2$, $x_4 = 1.4$, $x_6 = 1.6$, $x_8 = 1.8$, $x_{10} = 2.0$, $x_1 = 1.1$, $x_3 = 1.3$, $x_5 = 1.5$, $x_7 = 1.7$, $x_9 = 1.9$. We calculate the values of ordinates to seven decimal points (with an accuracy to within 0.00000005), and at once compile the necessary sums to substitute them into Simpson's formula. We receive: $y_0 = \frac{1}{x_0} = 1.0000000$, $y_{10} = \frac{1}{x_{10}} = 0.5000000$, $y_1 + y_{10} = 1.5000000$; $y_2 = \frac{1}{x_2} = 0.8333333$, $y_4 = \frac{1}{x_4} = 0.7142857$, $y_6 = \frac{1}{x_6} = 0.6250000$, $y_8 = \frac{1}{x_8} = 0.5555556$, $2(y_2 + y_4 + y_6 + y_8) = 5.4563492$; $y_1 = \frac{1}{x_1} = 0.9090909$, $y_3 = \frac{1}{x_3} = 0.7692308$, $y_5 = \frac{1}{x_5} = 0.6666667$, $y_7 = \frac{1}{x_7} = 0.5882353$, $y_9 = \frac{1}{x_9} = 0.5263158$, $4(y_1 + y_3 + y_5 + y_7 + y_9) = 13.8381580$.
Therefore, formula (6) gives for $\ln 2$ $(n = 5)$

$$\ln 2 = \int\limits_1^2 \frac{dx}{x} \approx \frac{1}{6 \cdot 5}(1.5000000 + 5.4563492 + 13.8381580) = 0.693150.$$

But using formula (*) on p. 43 we can compute $\ln 2$ with any degree of accuracy; we have only to assume $k = 1$, as was done on p. 43, and take n to be sufficiently large. Thus we can make sure that the value of $\ln 2$ corrected to eight decimal points is 0.69314718. Consequently, the value of $\ln 2$ obtained from Simpson's

formula differs from the true value by a number of the order of 0.000003, that is, the validity of this formula in this case is very great.

A more complicated analysis can demonstrate that Simpson's formula can yield a high degree of accuracy even for small n, when the graph of the function is very smooth and flat. The accuracy decreases when the graph contains very steep sections.

4. Let us apply Simpson's formula to calculate approximately the area of a circle. Since it is proportionate to the square of the radius, it is sufficient to carry out the calculations for a circle of radius equal to 1. Then, as we know, the area will be equal to $\pi \cdot 1^2 = \pi$. Hence our problem is to calculate approximately the number π using Simpson's formula. Using the property of the symmetry of a circle, we reduce the calculations to those of a quarter of the circle (Fig. 40). Then the result will be an approximate value of number $\frac{\pi}{4}$.

In this specific case we cannot expect a high degree of accuracy, although we make calculations for $n = 8$, for the reason of the very great steepness of the right side of the graph. Below we shall show what should be done in this case to improve the result. But now we shall begin the calculations. Since y is expressed in terms of x by the formula $y = \sqrt{1 - x^2}$ (Fig. 40), the problem reduces to

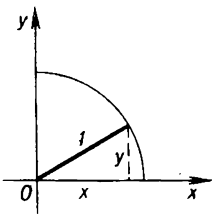

Fig. 40

computing the integral $\int_0^1 \sqrt{1 - x^2}\, dx$ by Simpson's formula. We shall assume $n = 8$. The abscissas of the point of division with even numbers will then pass through $\frac{1}{8}$, and those with odd numbers

will differ from the preceding points that are nearest to them by $\frac{1}{16}$. We obtain: $x_0 = 0$, $x_{16} = 1$, $x_2 = \frac{1}{8}$, $x_4 = \frac{1}{4}$, $x_6 = \frac{3}{8}$, $x_8 = \frac{1}{2}$, $x_{10} = \frac{5}{8}$, $x_{12} = \frac{3}{4}$, $x_{14} = \frac{7}{8}$, $x_1 = \frac{1}{16}$, $x_3 = \frac{3}{16}$, $x_5 = \frac{5}{16}$, $x_7 = \frac{7}{16}$, $x_9 = \frac{9}{16}$, $x_{11} = \frac{11}{16}$, $x_{13} = \frac{13}{16}$, $x_{15} = \frac{15}{16}$

Let us calculate the corresponding ordinates by the formula $y = \sqrt{1 - x^2}$ and complete the sums which are to be substituted into Simpson's formula. We receive: $y_0 = 1$, $y_{16} = 0.0000$, $y_0 + y_{16} = 1.0000$, $y_2 = 0.9922$, $y_4 = 0.9682$, $y_6 = 0.9270$, $y_8 = 0.8660$, $y_{10} = 0.7806$, $y_{12} = 0.6614$, $y_{14} = 0.4841$; $2(y_2 + y_4 + y_6 + y_8 + y_{10} + y_{12} + y_{14}) = 11.3590$; $y_1 = 0.9980$, $y_3 = 0.9823$, $y_5 = 0.9499$, $y_7 = 0.8992$, $y_9 = 0.8268$, $y_{11} = 0.7262$, $y_{13} = 0.5830$, $y_{15} = 0.3480$; $4(y_1 + y_3 + y_5 + y_7 + y_9 + y_{11} + y_{13} + y_{15}) = 25.2536$. It follows that

$$\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\, dx \approx \frac{1}{6 \cdot 8}(1.0000 + 11.3590 + 25.2536) = 0.7836.$$

When π is computed by other means with an accuracy to within 0.00005, the result is 3.1416, whence $\frac{\pi}{4} = 0.7854$. Hence the result obtained by Simpson's formula contains an error of the order of 0.002; it should be rounded off to 0.001: $\frac{\pi}{4} \approx 0.784$.

Now we shall use Simpson's formula with the same end in view (calculation of π) but in a more favourable situation. We shall move some distance from the steep right end of the graph and consider the integral $\int_0^{0.5} \sqrt{1 - x^2}\, dx$. If we take this integral as yielding the area of the curvilinear trapezoid $AOCD$ (Fig. 41) and subtract from it the area of the triangle OCD equal to $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 0.2165064$, we obtain the area of the circular sector AOD with the central angle $30° = \frac{360°}{12}$. Thus the difference

$\int_0^{0.5} \sqrt{1-x^2}\,dx - 0.2165064$ yields the value of one-twelfth of the area of the circle, i. e. $\frac{\pi}{12}$. Let us use Simpson's formula for $n = 4$ to compute $\int_0^{0.5} \sqrt{1-x^2}\,dx$; in this case the abscissas of the division points x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7 and x_8 will remain the same as before. But we shall now calculate the corresponding ordinates to seven decimal points to attain a high degree of accuracy in the result. In this way we receive the following values for the ordinates and their sums contained in Simpson's formula: $y_0 = 1.0000000$, $y_8 = 0.8660254$, $y_0 + y_8 = 1.8660254$; $y_2 = 0.9921567$, $y_4 = 0.9682458$, $y_6 = 0.9270248$; $2(y_2 + y_4 + y_6) = 5.7748546$; $y_1 = 0.9980450$, $y_3 = 0.9822646$, $y_5 = 0.9499178$, $y_7 = 0.8992184$, $4(y_1 + y_3 + y_5 + y_7) = 15.3177832$. Substituting these

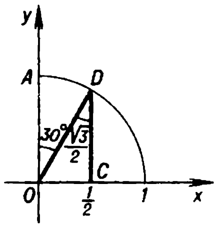

Fig. 41

values of the sums of the ordinates into Simpson's formula we obtain

$$\int_0^{0.5} \sqrt{1-x^2}\,dx \approx \frac{0.5}{6 \cdot 4}(1.8660254 + 5.7748546 + 15.3177832) = 0.4783055.$$

In accordance with the above reasoning it follows that

$$\frac{\pi}{12} = \int_0^{0.5} \sqrt{1-x^2}\,dx - 0.2165064 \approx 0.2617991.$$

The accuracy of the result obtained can be verified by multiplying it by 12 (thus increasing the error of the result the same number of times); this gives 3.1415892 as the value of the number π. But with an accuracy to within 0.0000005 the number π is equal to 3.141593 (this can be found by Simpson's formula for greater values of n; there exist, however, other computing techniques requiring less laborius calculations). Consequently, giving the result to only five decimal points we have $\pi \approx 3.14159$, with an accuracy of within 0.000005. This very close approximation to the famous number π was obtained by a skilful application of Simpson's formula.

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